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Axiomatic Foundations of Quantum Physics: Critiques and Misunderstandings. Piron's Question–Proposition System

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Some of the most frequent misconceptions about axiomatic quantum physics are discussed with the aim of clarifying their true significance, taking Piron's approach as conceptual framework. In particular, we deal with the following topics: the wrong identification of Piron's questions and Mackey's questions, and some curious alleged empirical consequences; the role of propositions as suitable equivalence classes of questions, their partial order structure, and the paradoxical consequences of the erroneous assignment to questions of some lattice properties involving propositions; the logical and the empirical purport of some "negative" theorems; the standard Hilbert space model of the theory and the consequent "metaphysical disasters" related to some identifications, which are peculiar of this model. A controversy between Foulis–Piron–Randall and Hadjisavvas–Thieffine–Mugur-Schächter is analyzed on the basis of the proposed Hilbert space model (in which Piron's questions are realized by Hilbertian "effects," i.e., linear bounded operators F such that $\mathbb{O} \leq F \leq \mathbb{1}$) which clarify the different point of views. As an example, we treat the unsharp localization operators in $L_2(\mathbb{R})$.

1. INTRODUCTION

This work is motivated by recent experiences of ours (private communications and discussions, referee reports, and so on) in which some misconceptions and misunderstandings about fundamental aspects of axiomatic quantum physics (QP) have been the cause of several and different, but in general not correct, critiques. Since these controversies are rather frequent and repeated, we have found it necessary to clarify some points, so that all the terms of the discussions about these arguments can be correctly formulated.

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In particular, we refer to the Jauch–Piron (JP) approach to the axiomatic foundations of QP presented in Piron, (1964, 1972, 1976*a*,*b*, 1977, 1978, 1981), Jauch (1968, 1971), and Jauch and Piron (1969, 1970). Like any *axiomatic formalized theory*, it is grounded on a system of *primitive notions*, the choice of a set of *specific axioms*, and some *interpretation rules* which translate the primitive notions into concepts about a physical domain. We emphasize that an axiomatic formalized theory is constructed on the basis of a *formal language* **L** in which a set of *signs* is precisely defined (in particular, *individual constant signs*, *individual variable signs*, *m-predicate signs*, and *n-functor signs*; all *derived notions*, *expressions*, and *sentences*, in particular all the axioms, of the theory will be finite sequences of the admitted language), without reference to any concrete mathematical structure (e.g., the one involved in the Hilbert space theory).

1.1. JP Question-Preparation Approach to QP

With reference to the JP approach, the concepts of "questions" and "preparations," with a binary relation of "question *true* in a preparation," are assumed as primitive notions of the theory.

We recall that, according to Piron (1976*a*), "we shall call a **question** every *experiment* leading to an alternative of which the terms are 'yes' or 'no'." It is worth noting that in Piron's view: "schematically a question consists of a *measuring apparatus*, instructions for its use and a rule interpreting the possible results in terms of 'yes' or 'no'" (Piron, 1977). In this sense one can speak of "physical" or "empirically definable" questions; no reference is made to any possible realization of the theory (e.g., to the standard orthodox Hilbert space model).

Thus, Piron's theory is based on the following statement:

> "When the physical system has been **prepared** in such a way that the physicist may affirm that in the event of an experiment the result 'yes' is *certain*, we shall say that the **question** is **true**" (Piron, 1976*b*).

Moreover, in another work JP stress that: "For the time being we are not concerned with the question how we can produce systems for which a given yes–no experiment is known to be '*true*' nor how we obtain this knowledge" (Jauch and Piron, 1969).

Therefore, a formal theory describing the JP approach to QP is based on the primitive notions of *preparation* and *question*, formally described by (mathematically uninterpreted) signs $x, y, \ldots$ and $\alpha, \beta, \ldots$, respectively, and a primitive binary predicate sign T involving preparation–question pairs; the formula $T(x, \alpha)$ is physically interpreted as "question α is *true* when the physical system is prepared in x."

One of the problems about any formal theory is its (relative) "coherence," which is solved if a concrete *mathematical model* of the theory is given ("the theory is coherent if mathematics is such"). This means that a *realization* of the language **L** on a concrete mathematical structure $\mathscr{H}$ must be given (in particular, individual constant and variable signs are *represented* as elements and variables in $\mathscr{H}$, *m*-predicate signs as *m*-argument concrete relations on $\mathscr{H}$, and *n*-functor signs as concrete mappings from $\mathscr{H}^n$ to $\mathscr{H}$), in such a way that *all* the axioms to the theory are validated in the model, i.e., their $\mathscr{H}$-realizations turn out to be *theorems* of the mathematical $\mathscr{H}$-structure.

It is possible to show that a realization of JP axiomatic theory, based on the mathematical structure of a complex Hilbert space $\mathscr{H}$, can be given (Cattaneo *et al.*, 1988). Precisely, preparations are represented by one-dimensional subspaces Φ of the Hilbert space $\mathscr{H}$; questions by linear, bounded, self-adjoint operators F on $\mathscr{H}$ such that $\mathbb{O} \le F \le \mathbb{1}$; the binary predicate T by the binary relation

$$\text{“}T_{\mathscr{H}}(\Phi, F) \quad \text{iff } \varphi\in\Phi,\ F\varphi=\varphi\text{”, (i.e., iff } \varphi\in\Phi/\{\underline{0}\},\ \langle F\varphi\,|\,\varphi\rangle/\|\varphi\|^2=1).$$

In this Hilbertian realization some "hidden" axioms and all Piron's C, P, A axioms are validated (i.e., are translated into statements which are theorems of the Hilbert space theory), concluding that this Hilbertian realization turns out to be a model of the JP approach to QP.

1.2. JP Properties and Mackey Questions, Related Misconceptions

In JP theory an important role is played by the partially ordered set of all propositions induced by the set Q of questions. A *proposition* is any equivalence class of questions with respect to the equivalence relation: "$\alpha, \beta\in Q$, $\alpha\sim\beta$ iff $T(x, \alpha) \Leftrightarrow T(x, \beta)$." Thus, the notion of proposition is a derived notion in the theory, and from the axioms it follows that the set of all such propositions $\mathscr{L} := \mathscr{Q}/\sim$ has a structure of atomic orthomodular complete lattice with covering property (structure named *propositional system* by Piron).

In our opinion, the source of many misconceptions is due to the following two facts. As regards JP theory, Piron (1976*a*) shows the following *Representation Theorem.*

Theorem. Every irreducible propositional system $\mathscr{L}$ of rank at least equal to 4 may be realized by a vector space V constructed on some division ring with involution $\mathbb{K}$ and endowed with a definite Hermitian form (generalized Hilbert space in Piron's terminology).

To be precise,

(P) The partially ordered set of all **propositions** in quantum mechanics is isomorphic to the partially ordered set of all closed subspaces of a generalized Hilbert space.

On the other hand, Mackey (1963) assumes the following **Axiom VII**:

(M) The partially ordered set of all **questions** in quantum mechanics is isomorphic to the partially ordered set of all closed subspaces in a separable, infinite-dimensional Hilbert space.

Hence, it is evident that if in Piron's Representation Theorem the ring $\mathbb{K}$ is the complex numbers $\mathbb{C}$ (and so V is a complex Hilbert space $\mathscr{H}$), *Mackey's questions are identifiable with the propositions of the JP approach* (which are equivalence classes of JP questions).

Table I sums up this discussion.

With respect to these considerations, we do agree with an extension of Foulis and Randall's point of view according to which orthodox (Hilbert space) quantum mechanics has encouraged people to some identifications peculiar to the Hilbert space mathematical structure, and this has been a "metaphysical disaster" (Randall and Foulis, 1983).

In our experience we have encountered some quite rough "metaphysical disasters," which can be summarized in the following types.

1. The lattice properties of propositions (owing to the identification with Mackey questions) are attributed to Piron's questions (which have no structure of a partially ordered set); consequently, a lot of curious, but in any case wrong, critiques have been produced (the "theoretical" collapse of the set of questions to the two trivial elements $\mathbb{0}$ and $\mathbb{1}$, the alleged invalidation of de Morgan's laws for this supposed lattice, and so on).

2. The empirical analysis of usual experimental situations (leading to the conclusion that there is no experimental yes–no device represented by

Table I. JP Properties[a]

	Jauch–Piron		$\mathscr{H}$-realization	
JP Questions	Q	$\xrightarrow{\mathscr{H}\text{-real}}$	$Q(\mathscr{H})$	
	$\downarrow^{\Phi}$		$\cup\mid$	
JP Propositions	$\mathscr{L}=(Q/\sim)$	$\xrightarrow{\mathscr{H}\text{-real}}$	$\mathscr{E}(\mathscr{H})\equiv\mathscr{M}(\mathscr{H})$	M Questions

[a] $Q(\mathscr{H})$ = Hilbert space linear operators such that $\mathbb{0}\leq F\leq\mathbb{1}$; $\mathscr{M}(\mathscr{H})$ = poset of closed subspaces of the Hilbert space $\mathscr{H}$; $\mathscr{E}(\mathscr{H})$ = corresponding poset of orthogonal projections; Φ = canonical mapping associating to any JP question the proposition generated by it.

orthogonal projections of standard orthodox quantum mechanics) and the identification between Piron's questions and Mackey's questions (represented by orthogonal projections) both lead to the conclusion that "empirically speaking, one probably has few or no (Piron) questions at all."

Some more refined critiques have been produced by Hadjisavvas *et al.* (1980), Thieffine, *et al.* (1981), and Thieffine (1983) based on some "negative" theorems; the arguments of these critiques have given rise to a subsequent controversy with Foulis and Randall (1984; Hadjisaves and Thieffine, 1984; see also Foulis *et al.*, 1983; Randall and Foulis, 1983). We will discuss in detail these points, too.

2. JP QUESTION-PREPARATION SYSTEM (THE "PHYSICAL SCHEME")

The approach of Jauch and Piron (JP) to quantum physics (QP), as presented mainly in Piron's papers, concerns a system of "questions" and "proposition" (henceforth, qp-s), characterized by the assent to the "program of realism," i.e., without making use of the notion of probability, according to the statement: "we avoid the notion of probability (· · ·). If one introduces probability at this stage of axiomatics one has difficulties of avoiding the criticism of Einstein that a state is not an attribute of an individual system but merely the statistical property of a homogeneous ensemble of similarly prepared identical systems" (Jauch and Piron, 1969).

To be precise, some *metatheoretical* statements, formulated as follows, are premised.

The *physical system.*

(MTI) By a physical system we mean a part of real world thought of as existing in space-time and external to the physicist.

The *program of realism.*

(MT2) (The aim of physics is) to give a complete description of each individual system as it is in all its complexity.

Having thus established an epistemological standpoint, the *basic concept* of "*question*" is introduced by observing that "the affirmation of the physicist in regard to a physical system are susceptible of being regulated by experiment. This control consists in general of a measurement the result of which is expressed by 'yes' or 'no' " (Piron, 1976*a*).

(BC1) We shall call a question every experiment leading to an alternative of which the terms are "yes" or "no".

Thus a question is both

(1) a description of an experiment to be carried out on the physical system considered;

(2) a rule enabling us to interprete the possible results in terms of "yes" or "no".

More schematically a question consists of

(i) a measuring apparatus,

(ii) instruction for its use,

(iii) a rule interpreting the possible result in terms of "yes" or "no" (Piron, 1977).

It is worth noting that in this view a question is an *experiment* consisting of a macroscopic *measuring apparatus*, objectively given and technically describable; in this sense one can speak of a "physical" or "empirically definable" question. The macroscopic arrangement measuring a question can interact with individual samples of the physical system in such a way "that a direct, objectively traceable [macroscopic alternative] effect occurs or does not occur (e.g., counter signal, a cloud-chamber-track, blackening of a photographic plate, etc.)" (Ludwig, 1971); "the presence of the effect is conventionally taken as the answer 'yes' whereas its absence is 'no' " (Mielnik, 1976). This point of view can be defined as the "*realism of the laboratory*." We stress that in this discussion we have considered experiments of *single test*, i.e., in which a *single* individual sample of the physical system under examination is tested by a certain apparatus (which measures a question) yielding as a result one of the two alternative answers, either "yes" or "no" ("If the result is 'yes,' the system has passed the *test*" (Piron, 1976*a*).)

A similar position can be found in K. Kraus:

> Another empirical fact is the existence of so called measuring instruments, which are capable of undergoing macroscopically observable changes due to ("triggered by") their interaction with single microsystems. The simplest type of measuring instrument is one on which just a single change may be triggered. For instance, an originally charged counter may be found either still charged or discharged, after it has been exposed to an electron (· · ·). (The result will depend, loosely speaking, on the efficiency of the counter, and on whether or not the electron "hits" it.) (· · ·) one usually defines the result of a single measurement to be "yes" if the effect occurs, and "no" if the effect does not occur. (· · ·) Assume now a [production of] a single microsystem, which then interacts with an observing apparatus, leading in turn either to the occurrence or the non-occurrence of the corresponding effect on the apparatus. Call this a "single experiment" (· · ·) (Krauss, 1983).

2.1. Three Physical Definitions

In the JP approach, the set of questions is equipped with an articulate structure according to some definitions (named by Piron):

(PD1) A *trivial* question exists which we denote as I, and which consists in nothing other than measuring anything (or doing nothing) and stating that the answer is "yes" each time.

(PD2) If α is a question, we denote by $\alpha^{\vee}$ the question, called the *opposite* or the *inverse* of α, obtained by exchanging the terms of the alternative. Thus, if the result of α (for an individual sample) is "yes," then that of $\alpha^{\vee}$ is "no" and vice versa. It is clear that $\alpha^{\vee}$ can be measured with the same physical equipment as that used for the measurement of α.

(PD3) If $\{\alpha_i\}$ is a family of questions, we denote by $\pi\alpha_i$ the question, called the *product*, defined in the following manner: (i) one chooses at random one of the α_i in the family (measuring apparatus), (ii) one performs the corresponding experiment (on an individual sample of the physical system) (instruction for its use), and (iii) one attributes to $\pi\alpha_i$ the answer thus obtained (a rule interpreting the results).

JP affirm that, as a consequence, one can prove the following results:

$$(\alpha^{\vee})^{\vee}=\alpha \qquad \text{and} \qquad (\pi\alpha_i)^{\vee}=\pi\alpha_i^{\vee} \tag{2.1}$$

2.2. A Binary Predicate between "Preparation" and "Questions"

Now, JP introduce the following definition of the binary predicate "*true*" involving the two notions of "*preparation*" and "*question*."

(BP1) "When the physical system has been **prepared** in such a way that the physicist may affirm that in the event of an experiment the result 'yes' is *certain*, we shall say that the **question** is '**true**' (Piron, 1976*b*).

We note that the word "prepared" in (BP1) implicitly introduces, besides the basic concept of question, the basic concept of preparation of physical objects.

(BC2) A *preparation* is realized by macroscopic apparatus which can produce both *single* individual samples and *ensembles* of individual samples under well-defined and repeatable conditions.

A question can be tested on each *sample* giving as a result of the measurement one of the two *alternative*, "yes" or "no" (*single test* of a question). To the *preparation* as a whole it is possible to attribute the *value* "true" if the result "yes" is certain (*elementary experiment* of a pair "preparation–question"). This occurs whenever "the samples of an ensemble *prepared* in the same way have given rise to the answer yes with certainty. In this case we have the right to claim that new single samples of the system *prepared* in the same way will give rise to the answer 'yes' " (Aerts, 1983).

Another statement by Piron clarifies very well the role of preparation apparatus:

> Let us suppose that we have a beam of photons. The experiment which consists in placing a polarizer in the beam defines a question. In fact it is possible to verify, by despatching photons one by one, that this experiment leads to a plain alternative: either a photon passes through, or it is absorbed. We shall define the [question α_ϕ] by specifying the orientation of the polarizer (the angle ϕ) and interpreting the passage of a photon as a "yes." Experience shows that, to obtain a photon **prepared** in such a way that "α_ϕ is true," it is sufficient to consider the photons which have traversed a first polarizer oriented at this angle [*preparation* x_ϕ!]. But experiment also shows that it is impossible to **prepare** photons [*preparation* $x_{\phi'}$, $\phi \neq \phi'$ (modulo π)] capable of traversing with complete certainty a polarizer oriented at the angle ϕ (· · ·) (Piron, 1976*a*).

2.3. An Improvement in JP qp-s

A further binary predicate between preparations and questions can be introduced which has not been considered by JP.

(BP2) When the physical system has been prepared in such a way that the physicist may affirm that in the event of an experiment the result "no" (resp., "yes") is certain for the question α (resp., $\alpha^{\vee}$), we shall say that the question α is "*false*."

The following rule is now a necessary consequence of the physical meaning attributed to (BP1) and (BP2) starting from the level of description of a single individual object.

(TF) There is no preparation with respect to which a question is simultaneously true and false.

In general, it may happen that some preparation could exist for a question such that this latter is neither true nor false; hence, a third binary predicate can be introduced.

(BP3) When the physical system has been prepared in such a way that in the event of an experiment neither the result "yes" nor the result "no" is certain, we shall say that the question is *indeterminate*.

In the orthodox JP approach, the predicate "false" and "indeterminate" are not explicitly considered and this has been a first source of misunderstanding. Indeed, we have two levels of description: the first one, which pertains to single individual samples, leading to the alternatives "yes" or "no" in executing a single test of a question; the second one, which pertains to preparations of individual samples, leading to the three alternative values, "true," "false," or "indeterminate" in executing a yes–no elementary experiment with any question.

According to binary predicates (BP1)–(BP3), physical definitions (PD1)–(PD3) must be modified into the following ones involving the second level only.

(A1) The trivial question I is always true (and so never false).
(A2) The question $\alpha^{\vee}$ is true whenever α is false and is false whenever α is true.
(A3) The question $\pi\alpha_i$ is true iff any α_i is true and is false iff any α_i is false.

2.4. JP Derived Definitions

Then, the following *theoretical definitions*, expressed in terms of (PD1)–(PD3) and (BR1), are introduced in the JP approach.

(DF1) The inverse question $I^{\vee}$, denoted by O, is called the *absurd* question.
(DF2) If the physical system is prepared in such a way that whenever α is true, one is sure that β is true, too, then the question α is said to be *stronger* or *less* than the question β and this is symbolized by $\alpha < \beta$.
(DF3) If two questions α and β satisfy relations $\alpha < \beta$ and $\beta < \alpha$, we shall call them JP *equivalent* and we denote it by $\alpha \sim \beta$. This relation is an equivalence relation.
(DF4) Let α be a question. We denote by $a = [\alpha]_{\sim}$ the class of all such questions which are JP equivalent to α and we call it a *proposition*. Thus $[\alpha]_{\sim} := \{\beta\colon \beta \sim \alpha\}$.
(DF5) The set of all propositions will be denoted by $\mathscr{L}$.
(DF6) If α is true, then every $\beta \sim \alpha$ is true, too. Hence we say that proposition a is *true* iff any (and therefore all) of $\beta \in a$ are true. If a proposition a is true, we shall call it a *property* of the system and we shall say that the system has *actually* the property a. So there is a one-to-one correspondence between propositions and properties.
(DF7) The equivalence classes of questions I and O define the *certain* and the *absurd* propositions **1** and **0**, respectively.
(DF8) If one has $\alpha \in a$, $\beta \in b$, and $\alpha \prec \beta$, then property a is *stronger* than property b, symbolized by $a \subseteq b$.

Remark 1. Whatever be the preparation, the certain property associated to proposition **1** is always actual, whereas the absurd property associated to proposition **0** is always potential.

2.5. FR Quasi-Ordering and FR Propositions

In our modified version of JP qp-s we can add the following further definitions.

(DF9) A question α is in *FR quasi-ordering* with question β, written $\alpha \prec \beta$, iff whenever α is true, one can affirm that β is true and whenever β is false, one can affirm that α is false.

(DF10) If two questions α and β satisfy relations $\alpha \prec \beta$ and $\beta \prec \alpha$, we shall call them *FR equivalent* and we denote it by $\alpha \simeq \beta$.

(DF11) Let α be a question. We denote by $p = [\alpha]_{\simeq} := \{\beta: \beta \simeq \alpha\}$ the class of all such questions which are FR equivalent to α and we call it an *FR proposition*.

(DF12) The set of all FR propositions will be denoted by $\mathscr{R}$.

(DF13) If α is true (resp., false) then every $\beta \simeq \alpha$ is true (resp., false), too. Hence we say that the FR proposition p is *true* (resp., *false*) iff any (and therefore all) of the $\beta \in p$ are true (resp., false).

(DF14) The FR equivalence classes of questions I and O define the *certain* and the *absurd* FR propositions $\mathbf{1}_{FR}$ and $\mathbf{0}_{FR}$, respectively.

(DF15) If one has $\alpha \in p$, $\beta \in q$, and $\alpha \prec \beta$, then the FR property p is *stronger* than the FR property q, symbolized by $p \preceq q$.

Since two FR equivalent questions are JP equivalent, too, any proposition is, in its turn, decomposable into a partition of FR propositions. Note that in the orthodox JP approach, binary predicates (BP2) and (BP3), and definitions (DF9)–(DF15) are not explicitly formulated.

2.6. JP States

We give now a definition of "state" according to the JP way of thinking:

(DF16) The *state* associated to a preparation procedure x is the set $\sigma(x)$ of all proposition [or properties, according to (DF6)] actually true [or certain] for the system prepared in x.

This definition matches with the following statement by Piron:

> If one given system has been prepared [according to a well-defined preparation procedure x] in such a way that we can affirm that in the event of the experiment [of a proposition a tested by any question $\alpha \in a$] the expected result would be certain, we will say that the corresponding property [associated to the proposition a] is an actual property of the system [prepared in x], in opposition to the other properties which [in x] are only potential (Piron, 1981).

In this case we have the right to claim that any single individual sample prepared in x actually possesses all the properties corresponding to the propositions of the state $\sigma(x)$, in agreement with the following statement: "a modified definition of state can be meaningfully applied to an individual system which represents all the properties (or elements of reality)" (Jauch and Piron, 1969).

Thus, the definition of state

> is meant to imply that the state is a property of an individual system and not of a statistical ensemble of such system. This was not possible in previous definitions of the state which involves probability (or probability amplitude). Indeed, a probability is meaningful only with reference to a statistical ensemble. The definition we have given above refers only to true propositions, that is to what we have called properties of the system, and there is no objection in attributing these properties to an individual system. (· · ·) We *attribute* to every system a state in the sense defined above quite independent whether the state has been measured (Jauch and Piron, 1969).

According to this point of view

> it has become possible to clarify the notion of state and that of physical property. The latter notion is closely related to that of *element of reality* introduced by Einstein, Podolsky and Rosen in the discussion of their paradox which bears their names. It is significant that these three authors come to the conclusion that the notion of state as used in quantum mechanics cannot meaningfully be attributed to an individual system and that it is a statistical concept, applicable only to suitably chosen assembly of systems (Jauch and Piron, 1969).

As to this assertion, we quote the following conclusion: "from this definition it is clear that according to Einstein's concept an *element of reality* is nothing else than an actual property" (Piron, 1981).

3. THE FORMALIZED THEORY OF JP QP-S (THE "SYNTACTIC SCHEME")

The formulation of JP qp-s outlined in the foregoing section is partially expressed in the colloquial physical language. We contend that between the two levels of description, the one relative to single individual samples and the one relative to preparations, the latter is more relevant from the physical point of view (statements about single individuals being either empirical or derived from statements of this second level). We have stressed that the JP approach to qp-s is founded on two primitive undefined kinds of notions, "preparations" and "questions," and a primitive undefined binary relation, "question *true* in a preparation." In this section, and referring to this second-level, we present an axiomatic formalized theory of JP qp-s giving the syntactic scheme of the JP approach to the foundation of quantum physics.

In order to avoid formal complications, but without any lost in correctness, we give a *formal prerealization* of the language of JP formalized theory based on the semiformal language of usual set theory and class logic. To be precise, the *language* of JP formalized theory consists of the following *alphabet*:

(a1) One kind of individual variable signs $x, y, z, \ldots$ (with indices, if necessary) realized as variables ranging over individuals in a nonempty abstract set S.

(a2) One kind of individual variable signs $\alpha, \beta, \gamma, \ldots$ (with indices, if necessary) realized as variables ranging over individuals in a nonempty abstract set Q.

(a3) One individual constant (0-argument functor) sign realized as a constant element I of Q; two functorial signs, a 1-argument functor ^{v} realized as a mapping $Q \mapsto Q$ and an infinitary-argument functorial sign π realized as a mapping $\mathscr{P}(Q) \mapsto Q$, where $\mathscr{P}(Q)$ is the power set of Q.

(a4) One 2-argument predicate sign realized as a binary relation $T \subseteq S \times Q$.

(a5) Usual classical logical connectives, $\wedge$, $\vee$, $\neg$, $\Rightarrow$, $\Leftrightarrow$, for conjunction, disjunction, negation, material implication, and biimplication, and the two quantifiers $\exists$ and $\forall$, the existential quantifier and the universal quantifier.

This alphabet of the JP qp-s will be denoted by

$$\mathbf{L}_{\mathrm{JP}} \equiv \langle S, Q, \mathscr{P}(Q); I, ^{v}, \pi; T \rangle$$

The set of all terms and of all wffs based on language $\mathbf{L}_{\mathrm{JP}}$ is constructed starting from the alphabet in the usual way. As customary, we will sometimes omit parentheses or employ other abbreviations in our wffs if there is no danger of confusion. As first derived notion, we define the 2-argument predicate sign

(D-BP2) $\quad F(x, \alpha) := T(x, \alpha^{v})$

The "rules of interpretations," which transform the formal language into a language about an empirical domain of physical objects, are the following:

(RI-1) The elements of the set S are interpreted as describing *preparations* of the physical system.

(RI-2) The elements of the set Q are interpreted as *questions* which one can be set up on the physical system.

(RI-3) The wff $T(x, \alpha)$ is physically interpreted as the statement "question α is *true* in preparation x" [i.e., (BP1)].

(RI-4) The wff $F(x, \alpha)$ is physically interpreted as the statement "question α is *false* in preparation x" [i.e., (BP2)].

We assume as *basic specific axioms* of JP qp-s the following wffs.

Axiom 1. $(\forall x), T(x, I)$
Axiom 2. $(\forall \alpha)(\forall x), (F(x, \alpha^{\vee}) \Leftrightarrow T(x, \alpha))$
Axiom 3. $(\forall \{\alpha_i\})(\forall x), (T(x, \pi\alpha_i) \Leftrightarrow (\forall \alpha_i) T(x, \alpha_i))$
$\wedge (F(x, \pi\alpha_i) \Leftrightarrow (\forall \alpha_i) F(x, \alpha_i))$
Axiom 4. $(\forall \alpha)(\forall x), \neg(T(x, \alpha) \wedge F(x, \alpha))$

Remark 1. Note that the (A1)–(A3) as modified versions of the "physical definitions" ((PD1)–(PD3)) and condition (TF) are the interpretations, via the above rules (RI-1)–(RI-4), of these "basic" specific axioms about JP qp-s theory. Moreover, it is possible to define the derived binary predicate "indeterminate" as follows:

(D-BP3) $\quad U(x, \alpha)$ iff $\neg(T(x, \alpha) \vee F(x, \alpha))$

For any fixed question $\alpha \in Q$, we introduce:

The *certainly-yes* domain of α: $S_1(\alpha) := \{x \in S: T(x, \alpha)\}$.
The *certainly-no* domain of α: $S_0(\alpha) := \{x \in S: F(x, \alpha)\}$.

Using these new definitions, the above axioms can be restated in the following form.

Axiom 1′. $S_1(I) = S$
Axiom 2′. $\forall \alpha, S_0(\alpha^{\vee}) = S_1(\alpha)$
Axiom 3′. $\forall \{\alpha_i\}, S_1(\pi\alpha_i) = \cap S_1(\alpha_i)$ and $S_0(\pi\alpha_i) = \cap S_0(\alpha_i)$
Axiom 4′. $\forall \alpha, S_1(\alpha) \cap S_0(\alpha) = \varnothing$

The formalized theory of JP qp-s is based on the following definitions.

(D1) (The absurd question) $\quad O := I^{\vee}$.
(D2) (JP quasi-order for questions) $\quad \alpha < \beta$ iff $T(x, \alpha)$ implies $T(x, \beta)$, i.e., $S_1(\alpha) \subseteq S_1(\beta)$.
(D3) (JP equivalent questions) $\quad \alpha \sim \beta$ iff $(\alpha < \beta)$ and $(\beta < \alpha)$, i.e., $S_1(\alpha) = S_1(\beta)$.
(D4) (Propositions) $\quad [\alpha]_{\sim} := \{\beta: \beta \sim \alpha\} = \{\beta: S_1(\beta) = S_1(\alpha)\}$.
(D5) (The set of all propositions) $\quad \mathscr{L} := \{[\alpha]_{\sim} : \alpha \in Q\}$; the elements of $\mathscr{L}$ are denoted by $a, b, c, \ldots$ (with indices, if necessary).
(D6) (Actual proposition) $\quad A(x, a)$ iff $(\alpha \in a)(T(x, \alpha))$.
(D7) (Trivial propositions) $\quad \mathbf{1} := [I]_{\sim}, \mathbf{0} := [O]_{\sim}$.
(D8) (Order relation for propositions) $\quad a \subseteq b$ iff $(\alpha \in a)$ and $(\beta \in b)$ imply $(\alpha < \beta)$.

By making use of the basic axioms and definitions above, JP prove the first fundamental theorem of the theory.

Theorem 3.1. The binary relaxation $\subseteq$ is a partial order on $\mathscr{L}$ with respect to which we denote, if they exist, by $\cap$ and $\cup$ and the g.l.b. and the l.u.b., respectively.

The set $\mathscr{L}$ of propositions is a complete lattice. In particular we have that:

(L1) Let $\{a_j\} \in \mathscr{P}(\mathscr{L})$; then

$$\cap\, a_j = [\pi \alpha_j]_{\sim} \qquad \text{and} \qquad \cup\, a_j = \cap\, \{c: (\forall j) a_j \subseteq c\}$$

(L2) $\mathbf{0} = \cap\, \{a: a \in \mathscr{L}\}$ and $\mathbf{1} = \cup\, \{a: a \in \mathscr{L}\}$

We do agree with Hadjisavvas *et al.* (HTM):

> The formalism briefly reproduced above is built on two interconnected levels, the level of questions and the level of propositions. For this reason we call it a *question-proposition system* and symbolize it by the notation qp-s. Now on the level of propositions, the logico-mathematical structure which emerges is that of a complete lattice [· · ·].
>
> The global structure introduced by the two levels, of propositions and of questions, interconnected according to Definitions (D1)–(D8) is a new structure, which involves more basic assertions than the usual lattice-theoretic formulations, namely those concerning relations between questions and propositions (Hadjisavvas *et al.*, 1980).

Quoting Aerts (1983), "Piron introduces the concept of 'question' to give a physical meaning to the concept of proposition that is used in quantum logic. He then introduces the lattice of properties of a physical system from this concept of question."

Remark 2. Referring to (L1), we quote the following assertion by Piron:

> the greatest lower bound of two propositions a and b has the following properties
>
> $$\text{“}a \cap b \textit{ true}\text{”} \Leftrightarrow \text{“}a \text{ true”} \text{ and } \text{“}b \text{ true”}$$
>
> which shows that $\cap$ plays the same role as "and" in logic. However, for the least upper bound, we have only
>
> $$\text{“}a \text{ true”} \text{ or } \text{“}b \text{ true”} \Rightarrow \text{“}a \cap b \text{ true”}$$
>
> In fact one has the following:
>
> *Proposition 3.1.* If "$a \cup b$ true" $\Leftrightarrow$ ("a true" or "b true") for all $a, b \in \mathscr{L}$, then $\mathscr{L}$ is distributive: $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$.
>
> In classical theory, $\mathscr{L}$ is the set of subsets of phase space and is distributive. The implication to the right in the theorem stated above is the essential distinction between classical and quantum theory (Piron, 1977).

One can add to definitions (D1)–(D8), which pertain to the orthodox JP approach, the following further definitions:

(D9) (FR quasi-order) $\alpha \prec \beta$ iff $T(x, \alpha)$ implies $T(x, \beta)$ and $F(x, \beta)$ implies $F(x, \alpha)$, i.e., $S_1(\alpha) \subseteq S_1(\beta)$ and $S_0(\beta) \subseteq S_0(\alpha)$.

(D10) (FR equivalent questions) $\alpha \simeq \beta$ iff $(\alpha \prec \beta)$ and $(\beta \prec \alpha)$, i.e., $S_1(\alpha) = S_1(\beta)$ and $S_0(\alpha) = S_0(\beta)$.

(D11) (FR propositions) $[\alpha]_{\simeq} := \{\beta : \beta \simeq \alpha\} = \{\beta : S_1(\beta) = S_1(\alpha)$ and $S_0(\beta) = S_0(\alpha)\}$.

(D12) (The set of all FR propositions) $\mathscr{R} := \{[\alpha]_{\simeq} : \alpha \in Q\}$; the elements of $\mathscr{R}$ are denoted by $p, q, r, \ldots$ (with indices, if necessary).

(D13) (Truth values) $\tau(x, p)$ iff $(\alpha \in p)(T(x, \alpha))$ and $\varphi(x, p)$ iff $(\alpha \in p)(F(x, \alpha))$.

(D14) (Trivial FR propositions) $\mathbf{1}_{\mathrm{FR}} := [I]_{\simeq}$, $\mathbf{0}_{\mathrm{FR}} := [0]_{\simeq}$.

(D15) (Order relation for FR propositions) $p \preceq q$ iff $(\alpha \in p)$ and $(\beta \in q)$ imply $(\alpha \prec \beta)$.

Remark 3. We recall that JP stated that the second equation of (2.1) can be easily derived. This derivation does not seem actually possible, as is also suggested by the fact that the physical arrangements corresponding to $(\pi\alpha_i)^{\vee}$ and $\pi\alpha_i^{\vee}$, according to rules (i)–(iii) of (PD3), do not exactly coincide. At any rate it is possible to state the following result:

$$\alpha^{\vee\vee} \simeq \alpha \qquad \text{and} \qquad (\pi\alpha_i)^{\vee} \simeq \pi(\alpha_i^{\vee})$$

Since the second of (2.1) is not needed to the development of JP theory, we will dispense with it in the sequel.

We intend to deal now with the notion of JP "state" and the related properties. To this end, we introduce the following definitions.

(D16) (JP state) $\sigma(x) = \{a \in \mathscr{L} : A(x, a)\}$.

(D17) (The set of all states) $\Sigma := \{\sigma(x) : x \in S\}$; the elements of Σ will be denoted by $u, v, w, \ldots$ (with indices, if necessary).

(D18) (JP pure state) is any JP state $\sigma(x)$ which is maximal in Σ.

(D19) (The set of all JP pure states) Σ_p is the set of all JP pure states.

(D20) (Pure preparation) is any preparation $x_p \in S$ whose associated JP state $\sigma(x_p)$ is pure (i.e., maximal in Σ).

(D21) We denote by S_p the set of all pure preparations; evidently,

$$\Sigma_p = \{\sigma(x_p) : x_p \in S_p\} = \sigma(S_p)$$

For any preparation $x \in S$ the set $\sigma(x)$ is the collection of all JP propositions which are *actually true*, i.e., *true with certainty*, in x. This set determines all the physical properties that can be attributed with certainty to the samples of the physical system prepared according to x, whether the state has been measured. Thus $\sigma(x)$ embodies the *amount of information* actually available for *any single sample* of the physical system prepared according to x. Hence, if $x_p \in S_p$ is a pure preparation, the information available on any individual sample prepared according to x_p is maximal, i.e., $\sigma(x_p)$ embodies a *maximal amount of information.*

The following theorem collects some basic properties of JP states (Jauch and Piron, 1969; Cattaneo *et al.*, 1989).

Theorem 3.2. The set Σ of all JP states satisfies the following properties.

(S1) If $a \in S(x)$ and $a \subseteq b$, then $b \in \sigma(x)$.
(S2) If $\{a_i: i \in I\} \subseteq \sigma(x)$, then $\bigcap_{i \in I} a_i \in \sigma(x)$.
(S3) $\mathbf{0} \notin \sigma(x)$, $\mathbf{1} \in \sigma(x)$ for every $\sigma(x)$.
(S4) For any $a \in \mathscr{L}$, $a \neq \mathbf{0}$, there exists at least one $\sigma(x)$ such that $a \in \sigma(x)$.

If for every $x \in S$ we put

$$e(x) = \bigcap_{a \in \sigma(x)} a$$

[which, owing to (S2), exists], then $e(x) \in \sigma(x)$ and

$$\sigma(x) = \{a \in \mathscr{L}: e(x) \subseteq a\}$$

Hence, we can conclude that $\sigma(x)$ is characterized by $e(x)$. Furthermore, let us denote by $\mathscr{L}_a$ the set of all atoms in lattice $\mathscr{L}$; whenever an atom $e \in \sigma(x) \cap \mathscr{L}_a$ exists, then $e = e(x)$;

Theorem 3.3. For every pure preparation $x_p \in S_p$, the proposition $e(x_p)$ characterizing the state $\sigma(x_p)$ is an atom of the lattice $\langle \mathscr{L}, \mathbf{0}, \subseteq \rangle$.

Conversely, for every atom e of the lattice $\langle \mathscr{L}, \mathbf{0}, \subseteq \rangle$ a pure preparation $x_p \in S_p$ exists such that $e(x_p) = e$.

It follows from Theorem 3.3 that every atom of $\mathscr{L}$ can be bijectively associated to a JP pure state according to the following one-to-one and onto correspondence:

$$\begin{gathered} \Sigma_p \equiv \mathscr{L}_a \\ \sigma(x_p) \leftrightarrow e(x_p) \end{gathered} \tag{3.1}$$

The existence of this one-to-one mapping allows us to identify any pure state with the corresponding atom. While the above definitions and results about JP states can be found in JP (see, for instance Jauch and Piron, 1969), the lack of any formalization of the notion of preparation neglects some important aspects, which we now complete.

(D22) (CGN equivalent preparations) $x \approx y$ iff $\sigma(x) = \sigma(y)$.
(D23) (CGN states) $[x]_{\approx} := \{y: y \approx x\} = \{y: \sigma(y) = \sigma(x)\}$.
(D24) (The set of all CGN states) $\mathscr{S} := \{[x]_{\approx} : x \in S\}$.
(D25) (JP state associated to a CGN state) $\sigma([x]_{\approx}) := \sigma(y)$, whatever be $y \in [x]_{\approx}$.

Any CGN equivalence class of preparations $[x]_{\approx} = \{y: \sigma(y) = \sigma(x)\}$ is identifiable with the unique JP state $\sigma(x)$, which is defined as the JP state $\sigma([x]_{\approx})$ of the whole equivalence class of preparations $[x]_{\approx}$. In symbols,

$$\begin{gathered} \Sigma \equiv \mathscr{S} \\ \sigma(x) \leftrightarrow [x]_{\approx} \end{gathered} \tag{3.2}$$

Thus, from now on, by a JP state we mean both the equivalence class $[x]_{\approx}$ and the set $\sigma([x]_{\approx})$ of all propositions true (properties actual) in this state.

(D26) (Properties actual in a state) $\mathscr{A}(w, a)$ iff $(x \in w)(\alpha \in a)$, $T(x, \alpha)$.

3.1. JP Peculiar Specific Axioms

Additional assumptions (*peculiar specific axioms* of JP qp-s) are introduced by JP "in order to recover (in some weak formulation) the usual mathematical Hilbert space structure of which the quantum theory makes technical use for describing the microworld" (Hadjisavvas *et al.*, 1980).

Axiom C. For each proposition $a \in \mathscr{L}$ there exists at least one compatible complement $b \in \mathscr{L}$, i.e., another proposition b such that:

(C1) (Complement) $a \cap b = \mathbf{0}$ and $a \cup b = \mathbf{1}$.
(C2) (Compatible) there exists a question $\gamma \in Q$ such that $\gamma \in a$ and $\gamma^{v} \in b$.

Remark 4. Axiom C, in part (C2), interconnects the two levels of description of a qp-s, the one relative to propositions (propositions a and b) and the one—this is very important—relative to questions (the existence of question γ).

Axiom P. Let $a, b \in \mathscr{L}$, and let a', b' be compatible complements of a, b, respectively; then, $a \subseteq b$ implies that the sublattice of $\mathscr{L}$ generated by $\{a, b, a', b'\}$ is Boolean.

Axiom A:
(A_1) Let $a \in \mathscr{L}$, $a \neq \mathbf{0}$; then, an atom p of $\mathscr{L}$ exists such that $p \subseteq a$.
(A_2) (Covering law). Let $a \in \mathscr{L}$, let p be an atom of $\mathscr{L}$, and let $a \cap p = \mathbf{0}$; then, $a \cup p$ covers a.

We now can quote the second fundamental JP theorem.

Theorem 3.4. In any qp-s in which Axioms C and P hold, we have the following results:

1. For every $a \in \mathscr{L}$ the compatible complement (whose existence is assured by Axiom C) is unique, and is denoted by $a' \in \mathscr{L}$ in the sequel).
2. The mapping

$$': a \in \mathscr{L} \to a' \in \mathscr{L}$$

is a standard orthocomplementation in $\mathscr{L}$.

3. With respect to this orthocomplementation, $\mathscr{L}$ is weakly modular (or orthomodular).

Summarizing, $\mathscr{L}$ is a complete, orthocomplemented, orthomodular lattice (i.e., a CROC in Piron's terminology). We recall that, according to Piron, a CROC for which Axiom A holds, i.e., which is atomic and satisfies the covering law, is said to be a *propositional system.*

4. THE "COHERENCE" OF JP QP-S (THE "HILBERTIAN MODEL")

Referring to qp-s, HTM stated that

> As far as we know, such a formal structure has not yet be studied. In particular it is not at all obvious *a priori* that this structure is formally self-consistent. (· · ·)
>
> In what follows we shall first show that the qp-s is self-consistent in the sense of the abstract theory of models, i.e., we shall show that it does admit a model. (· · ·) According to the theory of models a formal system is proved to be self-consistent if a model is produced for it, i.e., if a 'realization' of the language of this formal system is produced which validates all the axioms of the system. (· · ·).
>
> The conclusion imposed by [the existence of such a qp-s model] is far from being trivial, for two distinct reasons which may be related.
>
> (1) In the first place, as soon as one realizes fully the unusual character of the definition for a product question ($\pi\alpha_i$), the existence of at least a certain sort of self-consistency for the qp-s appears as surprising much more than natural.
>
> (2) In the second place, the model constructed [by HTM] possesses certain striking peculiarities; while the propositions are represented by closed subspaces of a Hilbert space, a product of questions is represented by a sum (of sets of subspaces).
>
> This suggests that the formal self-consistency proved with the help of such a drastic distorsion might somehow lead to difficulties in also mimicking the semantic structure associated to the quantum-mechanical formalism (Hadjisavvas *et al.*, 1980).

We quote the answer of Foulis and Randall (FR): "Thus we find it curious that the Hilbert space model selected by [HTM], to prove the self-consistency of qp-s (· · ·) was so complicated that their model-theoretic proof had to be relegated to an appendix because of its length. It seems even more curious that, even after such a laborious proof (· · ·)—establishing self-consistency *according to their own criterion*—they [i.e., HTM] could only grudgingly accord Piron's axioms 'a certain sort of self-consistency' " (Foulis and Randall, 1984). Moreover, FR assert that: "the book HTM are criticizing [Piron, 1976*a*] contains on p. 22 a simple and well known model establishing self-consistency for Piron's axiom system."

Remark 1. To tell the truth, on p. 22 Piron shows only an example of a propositional lattice which is not distributive, but this lattice is not a "model" for his axiom system. In particular, no assignment is given to represent questions and so it is not proved that this example validates Axiom C, in which both questions and propositions are involved.

Also, the following statement by Piron is not entirely correct, or, at least, if it is not correctly interpreted, could lead to misleading conclusions. "Let $\mathscr{L}$ be a CROC, i.e., a complete, orthocomplemented, and weakly modular lattice. *If one interprets the orthocomplement as a compatible complement*, then $\mathscr{L}$ satisfies Axioms C and P" (Piron, 1972).

Indeed, from a purely formal point of view, a CROC satisfies only Axioms (C1) and P, and not the (C2) part of Axiom C.

4.1. A Hilbert Space Model of qp-s

Above the HTM assertion, "in particular, it is not at all obvious *a priori* that this structure is formally consistent" must be understood in the sense that, at least to the best of our knowledge about JP, one does not find, before Hadjisavvas *et al.* (1980), a model of the JP axiom system.

In this section we give a model of a qp-s based on a complex (in general, separable) Hilbert space $\mathscr{H}$, quite different from that of HTM and summarized by the following concrete mathematical structure:

$$\mathbf{L}_{\mathrm{JP}}(\mathscr{H}) \equiv \langle S(\mathscr{H}), Q(\mathscr{H}), \mathscr{P}(Q(\mathscr{H})); \mathbb{1}, ', \Pi; T_{\mathscr{H}} \rangle$$

representing:

(i) *Preparation* by nonzero vectors ψ of the Hilbert space $\mathscr{H}$, i.e., $\psi \in S(\mathscr{H}) := \mathscr{H} / \{\underline{0}\}$.

(ii) *Questions* by linear operators on $\mathscr{H}$ such that $\mathbb{O} \leq F \leq \mathbb{1}$, i.e., $F \in Q(\mathscr{H})$ (Hilbert space "*effects*"). In particular, orthogonal projections on $\mathscr{H}$, whose collection will be denoted by $\mathscr{E}(\mathscr{H})$, are representatives of questions (Hilbert space "*exact* or *decision effects*").

(iii) The *trivial certain* question by the identity operator $\mathbb{1}$; the *inverse* question of F by $F' := \mathbb{1} - F$; the *product* of any family of questions $\{F_j : j \in J\}$ by the operator

$$\Pi F_j := \tfrac{1}{2}(E_{M_1(J)} + (E_{M_0(J)})')$$

where E_M is the orthogonal projection which projects onto the subspace M and

$$M_1(J) := \bigcap_{j \in J} \mathrm{Ker}(\mathbb{1} - F_j) \qquad \text{and} \qquad M_0(J) := \bigcap_{j \in J} \mathrm{Ker}(F_j)$$

(iv) Predicate of *question true* in *a preparation* by the following relation:

(H.BP1) $T_{\mathscr{H}}(\psi, F)$ iff $\psi \in \mathrm{Ker}(\mathbb{1} - F)/\{\underline{0}\}$

[Note that $\psi \in \mathrm{Ker}(\mathbb{1} - F)$ iff $F\psi = \psi$ iff $\langle F\psi \mid \psi \rangle = \|\psi\|^2$.]

In accordance with the general JP affirmation (2.1), it is trivial to prove that:

(H.1) $F = (F')$ and $(\Pi F_j)' = \Pi_j(F_j')$.

Moreover, we define the following binary relation:

$$\begin{aligned} F_{\mathscr{H}}(\psi, F) \quad &\text{iff} \quad T_{\mathscr{H}}(\psi, F') \\ \text{(H.BP2)} \qquad &\text{iff} \quad \psi \in \mathrm{Ker}(F)/\{\underline{0}\} \end{aligned}$$

[Note that $\psi \in \mathrm{Ker}(F)$ iff $F\psi = \underline{0}$ iff $\langle F\psi \mid \psi \rangle = 0$.]

Remark 2. The above binary relations "$T_{\mathscr{H}}$" and "$F_{\mathscr{H}}$," which are the Hilbertian representatives of the binary predicates "true" and "false" of JP qp-s, can be restated in the following way ($\psi \neq \underline{0}$):

(H.T) $T_{\mathscr{H}}(\psi, F)$ iff $\langle F\psi \mid \psi \rangle / \|\psi\|^2 = 1$.
(H.F) $F_{\mathscr{H}}(\psi, F)$ iff $\langle F\psi \mid \psi \rangle / \|\psi\|^2 = 0$.

These are the translations, inside our Hilbert space realization, of the statements "question F is true in preparation ψ" and "question F is false in preparation ψ," respectively, and we contend that, contrary to the HTM Hilbert space model, no "difficulties in also mimicking the semantical structure associated to the quantum-mechanical formalism" are produced by the above Hilbertian interpretation of JP qp-s.

Indeed, according to (H.T) and (H.F), formula $T_{\mathscr{H}}(\psi, F)$ [resp., $F_{\mathscr{H}}(\psi, F)$] agrees with the usual interpretation in the orthodox Hilbert space quantum mechanics, "in preparation ψ the probability of occurrence of effect F is 1 (resp., 0), i.e., effect F occurs (resp., does not occurs) with certainty in state ψ."

Let $F \in Q(\mathscr{H})$, for the sake of simplicity; in the sequel we set

$$M_1(F) := \mathrm{Ker}(\mathbb{1} - F) \qquad \text{and} \qquad M_0(F) := \mathrm{Ker}(F)$$

and we introduce the following notations:

The *certainly-true domain* of F: $S_1(F) := M_1(F)/\{\underline{0}\}$.
The *certainly-false domain* of F: $S_0(F) := M_0(F)/\{\underline{0}\}$.

The following results can easily be proved, and immediately imply that the basic specific axioms of JP qp-s are validated.

Theorem 4.1:
(H.A1) $S_1(\mathbb{1}) = S(\mathscr{H})$.
(H.A2) $S_0(F') = S_1(F)$.
(H.A3) $S_1(\Pi F_j) = \cap S_1(F_j)$ and $S_0(\Pi F_j) = \cap S_0(F_j)$.
(H.A4) $S_1(F) \cap S_0(F) = \varnothing$.

Definitions (D1)–(D8) of Section 3 are translated into the following Hilbertian definitions, respectively:

(H.D1) $\mathbb{O} = \mathbb{1}'$.
(H.D2) $F_1 < F_2$ iff $(F_1 \psi = \psi) \Rightarrow (F_2 \psi = \psi)$.
(H.D3) $F_1 \sim F_2$ iff $(F_1 \psi = \psi) \Leftrightarrow (F_2 \psi = \psi)$.
(H.D4) $[F]_\sim := \{G: S_1(F) = S_1(G)\} = \{G: M_1(F) = M_1(G)\}$.

Therefore, all questions of a given *proposition* are characterized by the same certainly-true domain; this common certainly-true domain turns out to be the *satisfaction domain* of the proposition. Hence, setting

(H.D5) $\mathscr{L}(\mathscr{H}) = \{[F]_\sim : F \in Q(\mathscr{H})\}$

there is a one-to-one correspondence between JP propositions of the Hilbertian model and subspaces of the Hilbert space and, as a consequence of Hilbert space theory, between JP propositions and orthogonal projections:

$$\begin{array}{c} Q(\mathscr{H}) \\ \cup| \\ \mathscr{L}(\mathscr{H}) \equiv \mathscr{M}(\mathscr{H}) \equiv \mathscr{E}(\mathscr{H}) \\ [F]_\sim \leftrightarrow M_1(F) \leftrightarrow E_{M_1(F)} \end{array} \tag{4.1}$$

Continuing the discussion of our Hilbert space model, we get

(H.D6) $A_{\mathscr{H}}(\psi, [F]_{\sim})$ iff $G \in [F]_{\sim}$ and $G(\psi) = \psi$

or, equivalently, if we identify, according to (4.1), Hilbertian JP propositions from $\mathscr{L}(\mathscr{H})$ with orthogonal projections from $\mathscr{E}(\mathscr{H})$,

(H.D6′) $A_{\mathscr{H}}(\psi, E)$ iff $\psi \in S_1(E)$.

Lastly,

(H.D7) $\mathbf{1} = [\mathbb{1}]_{\sim} = \{\mathbb{1}\}$
$\mathbf{0} = [\mathbb{O}]_{\sim} = \{G_0\colon G_0(\psi) = \psi \text{ iff } \psi = \underline{0}\}$

(H.D8) $[F]_{\sim} \subseteq [G]_{\sim}$ iff $M_1(F) \subseteq M_1(G)$.

Remark 3. One can also easily construct the Hilbertian representations of definitions (D9)–(D15). Anyway, we will omit them since they are not essential for the future developments.

We discuss now the Hilbertian representations of JP states and CGN states, where, for every Hilbertian preparation $\psi, \varphi, \ldots,$ we denote by $\Psi, \Phi, \ldots,$ the one-dimensional closed subspaces generated by them:

(H.D16) $\sigma(\psi) := \{E \in \mathscr{E}(\mathscr{H})\colon \psi \in S_1(E)\}$.

Remark 4. Note that, if we denote by E_Ψ the orthogonal projection which projects onto the one-dimensional subspace Ψ of $\mathscr{H}$, we have that $S_1(E_\Psi) = \Psi/\{\underline{0}\}$ and so E_Ψ belongs to the JP state $\sigma(\psi)$.

Since any E_Ψ is an atom of the Hilbertian propositional lattice $\mathscr{L}(\mathscr{H})$, we have that all Hilbertian JP states $\sigma(\psi)$ are pure states [owing to Theorem 3.3 and according to (D18)], and all Hilbertian preparations $\psi \in S(\mathscr{H})$ are pure preparations [according to (D20)]. We denote by $\Sigma_p(\mathscr{H})$ the set of all pure Hilbertian JP states, in the sequel.

(H.D22) $\varphi \approx \psi$ iff $\Phi = \Psi$; i.e., two Hilbertian preparation procedures are CGN equivalent iff they generate the same one-dimensional subspace of $\mathscr{H}$.

(H.D23) and (H.D24) $[\psi]_{\approx} \equiv \Psi/\{\underline{0}\}$; i.e., Hilbertian CGN pure states are identifiable with one-dimensional subspaces of $\mathscr{H}$, whose collection will be denoted by $\mathscr{S}_p(\mathscr{H})$.

Summarizing the results about states in the Hilbertian model of JP qp-s [and taking into account identifications (3.1) and (3.2)], we can get the following graph:

$$\begin{array}{ccccc} \Sigma_p(\mathscr{H}) & \equiv & \mathscr{L}_a(\mathscr{H}) & \equiv & \mathscr{S}_p(\mathscr{H}) \\ \sigma(\psi) & \leftrightarrow & E_\Psi & \leftrightarrow & \Psi \end{array} \tag{4.2}$$

We stress that JP pure states are, in this Hilbertian model, identifiable with one-dimensional subspaces which, in their turn, are identifiable with the corresponding orthogonal projection.

Theorem 4.2. In the above Hilbertian realization of JP qp-s, the following holds.

Condition C. For each proposition $a=[F]_\sim$, whose certainly-true domain is identifiable with subspace $M_1(a)=\mathrm{Ker}(\mathbb{1}-F)=M_1(F)$, there exists the (unique) compatible complement $a'=[E_{M_1(a)^\perp}]_\sim$, whose certainly-true domain is identifiable with subspace $M_1(a')=M_1(a)^\perp$. In particular:

(H.C2) There exists exactly one question, precisely $E_{M_1(a)}$, such that

$$(E_{M_1(a)})\in a \qquad \text{and} \qquad (E_{M_1(a)})'\in a'$$

Moreover:

(i) Proposition a contains a lot of questions, i.e., all the questions represented by operators $\mathbb{0}\leq G\leq\mathbb{1}$ such that $\mathrm{Ker}(\mathbb{1}-G)=M_1(a)$.

(ii) Proposition a' contains a lot of questions, i.e., all the questions represented by operators $\mathbb{0}\leq G_0\leq\mathbb{1}$ such that $\mathrm{Ker}(\mathbb{1}-G_0)=M_1(a')=M_1(a)^\perp$.

(iii) For any $F\in a$ we can construct proposition $c=[F']_\sim$, but in general $c\neq a'$, precisely,

$$\forall F\in a \quad \text{with} \quad F\neq E_{M_1(a)} \quad \text{we have} \quad F'\notin a'=[E_{M_1(a)^\perp}]_\sim$$

(iv) Question $E_{M_1(a)}$ minimizes the randomness of the certainly-no domains of all other $G\in a$; i.e., we get

$$M_1(a')=M_0(E_{M_1(a)})=\bigcup_{G\in a} M_0(G)$$

Therefore, this Hilbertian realization of the qp-s validates Axiom C and there is no difficulty to prove that it also validates Axioms P and A [we use the convention of naming "*quantum mechanics*" (henceforth, QM) the Hilbert space model, with the above outlined "interpretations," of the axiomatic formalized theory of QP according to the JP approach to qp-s).

In conclusion, we can assert the following:

Theorem 4.3. $\mathscr{T}_1$. The qp-s admit a model (based on the Hilbert space mathematical structure).

In our JP qp-s model [see (4.1)] the "*propositional logic*" $\mathscr{L}(\mathscr{H})$ of all Hilbertian propositions is identifiable with the propositional systems (i.e., orthomodular orthocomplemented atomic complete lattices satisfying the covering condition) $\mathscr{M}(\mathscr{H})$, of all closed subspaces of $\mathscr{H}$, and $\mathscr{E}(\mathscr{H})$, of all orthogonal projections on $\mathscr{H}$.

Since $\mathscr{E}(\mathscr{H}) \subseteq Q(\mathscr{H})$, we have the following conclusions: (1) any orthogonal projection E is a Hilbertian realization of some question, (2) any JP Hilbertian Proposition a is generated by a unique orthogonal projection, and, (3) owing to (iv), Theorem 4.2, this orthogonal projection "minimizes the randomness of the certainly-no domains" (Mielnik, 1976) of all other Hilbertian questions from a.

Hence, in axiomatic QM, it is customary to assume that orthogonal projections represent yes–no measuring apparatus which *sharply*, i.e., without noise and imprecision, test the corresponding Hilbertian JP propositions.

This is the reason people identify any proposition with the experimental sharp yes–no device which tests it, and, paraphrasing an FR assertion, "for an orthodox (Hilbert space) quantum mechanical entity $(\cdots)$, the canonical mapping $\mathscr{E}(\mathscr{H}) \mapsto \mathscr{L}(\mathscr{H})$ is a lattice isomorphism and both $\mathscr{E}(\mathscr{H})$ and $\mathscr{L}(\mathscr{H})$ are isomorphic to the lattice of closed subspaces of the Hilbert space. Although this is mathematically quite convenient, it has encouraged people to identify $\mathscr{E}(\mathscr{H})$ with $\mathscr{L}(\mathscr{H})$ and this has been a metaphysical disaster" (Randall and Foulis, 1983).

If this identification has been a source of miscomprehensions, another source of miscomprehensions, more subtle, could rise from the identification (4.2) between Hilbertian JP (or CGN) pure states and projections onto one-dimensional subspaces, which in any case are representatives of a certain class of sharp Hilbertian questions. A careless identification of $\Sigma_p(\mathscr{H})$ [or $\mathscr{S}_p(\mathscr{H})$] with this class would lead to another metaphysical disaster.

4.2. An Example: Localization in $\mathbb{R}$

First, we outline a JP qp-s for the *classical localization* of a particle in a one-dimensional space, say described by the real line $\mathbb{R}$. In this realization we represent:

(i) *Classical preparation procedures* by elements $x \in \mathbb{R}$.

(ii) *Classical localization questions* by Borel functions $\nu: \mathbb{R} \mapsto [0, 1]$, whose collection is denoted by $Q(\mathbb{R})$. For any $\nu \in Q(\mathbb{R})$ we introduce the Borel subsets

$$\Delta_1(\nu) := \nu^{-1}(\{1\}) \qquad \text{and} \qquad \Delta_0(\nu) := \nu^{-1}(\{0\})$$

In particular, characteristic functions χ_Δ of Borel subsets Δ of $\mathbb{R}$, whose set is denoted by $\mathscr{E}(\mathbb{R})$, are representatives of classical localization questions.

(iii) The *classical certain localization* question by the function $\underline{1}$ such that, $\forall x \in \mathbb{R}$,

$$\underline{1}(x) := 1$$

The *inverse* of v (which is a classical localization question, too) by $v' := \underline{1} - v$ (note that $\chi'_{\Delta} = \chi_{\Delta^c}$).

The *product* of any family $\{v_j : j \in J\}$ by the function

$$\pi\{v_j\} := \tfrac{1}{2}(\chi_{\Delta_1(J)} + \chi'_{\Delta_0(J)})$$

where

$$\Delta_1(J) := \bigcap_{j \in J} \Delta_1(v_j) \qquad \text{and} \qquad \Delta_0(J) := \bigcap_{j \in J} \Delta_0(v_j)$$

(iv) Binary predicate of "localization *true* in a preparation" by the binary relation

(C.BP1) $\quad T_c(x, v)$ iff $v(x) = 1$.

Binary predicate of "localization *false* in a preparation" is represented by the binary relation

(C.BP2) $\quad F_c(x, v)$ iff $v(x) = 0$.

The certainly-true and certainly-false domains of v are, respectively,

$$S_1(v) := \{x \in \mathbb{R} : T_c(x, v)\} = \Delta_1(v)$$

$$S_0(v) := \{x \in \mathbb{R} : F_c(x, v)\} = \Delta_0(v)$$

Example 4.1. If v_1 and v_2 are two exact classical localization questions whose certainly-true domains are intervals $\Delta_1(v_1) = (1, 5)$ and $\Delta_1(v_2) = (3, 8)$, respectively, then their product is the classical localization question

$$\pi\{v_1, v_2\}(x) = \begin{cases} 1, & x \in (3, 5) \\ 1/2, & x \in (1, 3] \cup [5, 8) \\ 0, & \text{otherwise} \end{cases}$$

for which $\Delta_1(\pi\{v_1, v_2\}) = (3, 5)$ and $\Delta_0(\pi\{v_1, v_2\}) = (-\infty, 1] \cup [8, \infty)$.

Without going into details, we remark that a classical localization proposition is the equivalence class of all classical localization questions having the same certainly-true domain. Therefore, any classical localization proposition is characterized by a Borel set Δ, the certainly-yes domain common to all classical localization questions of the proposition, and represents the classical property $l_c(\Delta) =$ "the particle is (classically) localized in Δ." This property is *sharply* measured by the unique exact classical localization question $\chi_\Delta \in l_c(\Delta)$, whereas all the other classical localization questions $v \in l_c(\Delta)$ give an *unsharp* measurement of the same property.

Quantum Description of Localization in $\mathbb{R}$

Let $L_2(\mathbb{R})$ be the Hilbert space of the quantum description of a particle in a one-dimensional space. For every Borel measurable function $v\colon \mathbb{R} \mapsto [0, 1]$, i.e., classical localization question $v \in Q(\mathbb{R})$, we introduce the linear operator $F(v)\colon L_2(\mathbb{R}) \mapsto L_2(\mathbb{R})$ defined, for any $f \in L_2(\mathbb{R})$, as follows:

$$(F(v)f)(x) := v(x) f(x)$$

Evidently, according to the Hilbertian model of Section 4.1, $F(v) \in Q(L_2(\mathbb{R}))$ is a Hilbertian question whose inverse $F(v)' \in Q(L_2(\mathbb{R}))$ is the question

$$(F(v)'f)(x) := (1 - v(x)) f(x) = v'(x) f(x)$$

The certainly-yes and certainly-no domains of $F(v)$ are

$$S_1(F(v)) = \{\psi \in L_2(\mathbb{R})/\{\underset{\sim}{0}\} : \operatorname{supp}(\psi) \subseteq \Delta_1(v)\} \tag{4.3}$$

$$S_0(F(v)) = \{\varphi \in L_2(\mathbb{R})/\{\underset{\sim}{0}\} : \operatorname{supp}(\varphi) \subseteq \Delta_0(v)\} \tag{4.4}$$

Remark 5. Note that for any pair of classical localization questions $v_1, v_2 \in Q(\mathbb{R})$, condition

$$\Delta_1(v_1) \cap \Delta_1(v_2) = \varnothing \quad \text{implies} \quad M_1(F(v_1)) \perp M_1(F(v_2))$$

Moreover, if for a classical localization question $v \in Q(\mathbb{R})$ both $\Delta_1(v)$ and $\Delta_0(v)$ are not empty, then in any preparation procedure $f \in S(L_2(\mathbb{R}))$, with $\operatorname{supp}(f) = \mathbb{R}$, the Hilbertian question $F(v)$ is neither "true" nor "false."

To any classical exact localization question χ_Δ, which sharply measured the classical property $l_c(\Delta) =$ "the particle is (classically) localized in Δ," there corresponds the Hilbertian exact localization question $E(\Delta) := F(\chi_\Delta)$ defined as

$$(E(\Delta)f)(x) := \chi_\Delta(x) f(x)$$

Remark 6. Note that $E(\Delta)' = E(\Delta^c)$.

Moreover, for any family $\{F(v_j) : j \in J\}$ of the Hilbertian localization questions, their "product" is the Hilbertian localization question

$$\begin{aligned}\Pi\{F(v_j)\} &= \frac{1}{2}\left(E\left(\bigcap_{j\in J} \Delta_1(v_j)\right) + E\left(\bigcap_{j\in J} \Delta_0(v_j)\right)'\right) \\ &= \tfrac{1}{2}(\chi_{\Delta_1(J)} + \chi_{(\Delta_0(J))^c})(\cdot)\end{aligned} \tag{4.5}$$

Two Hilbertian localization questions $F(\nu_1)$ and $F(\nu_2)$ are JP equivalent iff $\Delta_1(\nu_1) = \Delta_1(\nu_2)$. Then any Hilbertian localization proposition is characterized by a Borel subset Δ and is associated to the Hilbertian property $l_q(\Delta)$ = "the particle is (quantistically) localizated in Δ." Hence,

$$l_q(\Delta) \equiv \{F(\nu) : \Delta_1(\nu) = \Delta\} = [E(\Delta)]_{\sim} \qquad (4.6)$$

Property $l_q(\Delta)$ is sharply tested by the Hilbertian question represented by the orthogonal projection $E(\Delta)$; any Hilbertian localization question $F(\nu)$, for which $\Delta_1(\nu) = \Delta$, measures in an unsharp way the same property.

Orthogonal projection $E(\Delta)$ projects onto the subspace of all $f \in L_2(\mathbb{R})$ whose support is in Δ and so these nonzero vectors f, according to (H.D6′), represent preparation procedures in which this property is actual (i.e., true with certainty); each single individual sample prepared according to any of these preparation procedures gives the answer "yes" to property $l_q(\Delta)$.

Orthogonal projection $E(\Delta)'$ projects onto the subspace of all $g \in L_2(\mathbb{R})$ whose support is in Δ^c. Hence, according to Theorem 4.2, if $l_q(\Delta) \equiv [E(\Delta)]_{\sim}$ is the proposition "the particle is (quantistically) localized in Δ," then

$$l_q(\Delta)' \equiv [E(\Delta)']_{\sim} = [E(\Delta^c)]_{\sim} \equiv l_q(\Delta^c) \qquad (4.7)$$

That is, the proposition "the particle is not (quantistically) localized in Δ" is just the proposition "the particle is (quantistically) localized in Δ^c." Each nonzero vector from $L_2(\mathbb{R})$ whose support is contained in Δ^c represents a preparation procedure of single individual samples for which property $l_q(\Delta)'$ is true with certainty.

We remark that for any pair of nonempty, mutually disjoint Borel sets, $\Delta' \cap \Delta'' = \varnothing$, the corresponding exact localization questions $E(\Delta')$ and $E(\Delta'')$ project onto two subspaces $M_1(\Delta')$ and $M_1(\Delta'')$, which are mutually orthogonal. Now, the property "the particle is (quantistically) localized in $\Delta' \cup \Delta''$" is sharply measured by the exact localization question $E(\Delta' \cup \Delta'') = E(\Delta') + E(\Delta'')$ which projects onto subspace $M_1(\Delta') + M(\Delta'')$. If $\psi' \in M_1(\Delta')/\{\underline{0}\}$ [resp., $\psi'' \in M_1(\Delta'')/\{\underline{0}\}$], i.e., is a preparation procedure of single-particle localized in Δ' (resp., Δ'') with certainty, then $\psi' + \psi'' \in M_1(\Delta' \cup \Delta'')/\{\underline{0}\}$ is a preparation procedure of particle localized in $\Delta' \cup \Delta''$ with certainty,

$$T_q(\psi' + \psi'', E(\Delta' \cup \Delta''))$$

However,

$$\neg T_q(\psi' + \psi'', E(\Delta')) \qquad \text{and} \qquad \neg T_q(\psi' + \psi'', E(\Delta''))$$

In conclusion, the preparation $\psi' + \psi''$ is such that the property "the particle is (quantistically) localized in $\Delta' \cup \Delta''$" is "true" with certainty, but in which we cannot state either that "the particle is (quantistically) localized in Δ'" or that "the particle is (quantistically) localized in Δ''" is true.

5. MACKEY QUESTIONS, JP QUESTIONS, AND RELATED MISUNDERSTANDINGS

The Hilbert space model of JP qp-s previously introduced can be summarized in the following scheme:

$$\begin{array}{ccc} \alpha \in Q & \xrightarrow{\mathscr{H}\text{-real}} & F \in Q(\mathscr{H}) \\ \downarrow & & \downarrow \\ [\alpha]_{\sim} \in \mathscr{L} & \xrightarrow{\mathscr{H}\text{-real}} & [F]_{\sim} \in \mathscr{L}(\mathscr{H}) \equiv \mathscr{M}(\mathscr{H}) \equiv \mathscr{E}(\mathscr{H}) \end{array}$$

We recall the following definitions, which compell us to make a clear distinction between "questions" of the JP approach and "questions" of the Mackey (M) approach. First we have the following definition pertaining to the JP approach.

JP Questions. "A **question** is every experiment leading to an alternative of which the terms are 'yes' or 'no'" (the set Q).

As recalled in the Introduction, Piron's isomorphism theorem can be formulated in the following way.

(P) The partially ordered set of all **propositions** in quantum mechanics is isomorphic to the partially ordered set of all closed subspaces of a vector space $\mathscr{V}$, constructed on some division ring with involution $\mathbb{K}$ and endowed with a nondegenerate sesquilinear form $\langle\,|\,\rangle : \mathscr{V} \times \mathscr{V} \mapsto \mathbb{K}$.

The role of specific peculiar axioms in the JP approach is pointed out by Aerts (1983) in the following statement: "He [Piron] also defines a set of axioms on this lattice, such that when these axioms are satisfied, the theory becomes a theory equivalent to quantum mechanics (in Hilbert space). $(\cdots)$ Of course, his aim was to clarify quantum mechanics, and therefore he was looking for a set of axioms that would reduce the a priori more general theory to a theory as quantum mechanics."

On the other hand, we have the following assumption in the Mackey approach.

M Questions. "The partially ordered set of all **questions** in quantum mechanics is isomorphic to the partially ordered set of all closed subspaces of a separable, infinite dimensional Hilbert space" [Axiom VII of Mackey (1963)], [the set $\mathscr{M}(\mathscr{H}) \equiv \mathscr{L}(\mathscr{H})$].

Therefore, we must distinguish between:

(a) $\mathcal{Q}$: JP-questions.
(b) $\mathcal{L} \equiv \mathcal{M}(\mathcal{H})$: JP-propositions, identified with M questions.

With respect to this, we stress that the structure $\langle Q; I, \pi, ^v \rangle$ is not a poset (and so neither a lattice). Hence, π cannot be considered as a lattice *meet* with respect to some supposed (or imagined) partial ordering, nor is ^{v} a sort of *orthocomplementation*.

On the contrary, $\langle \mathcal{L}; \mathbf{1}, \cap, ' \rangle$ is an orthocomplemented atomic complete lattice in which $\cap$ is the lattice meet with respect to a well-defined ordering and $'$ is a standard orthocomplementation mapping on this poset.

The identification between M questions and JP questions is the source of many misconceptions, which can be based on the following HTM statement: "The definition of a product of questions and the rule for the 'negation' of such a product

$$(\pi\alpha_j)^v = \pi_j(\alpha_j^v)$$

are unusual and seem to contradict de Morgan's law" (Hadjisavvas *et al.*, 1980).

This doubt about the contradiction of de Morgan's law is explicit in the following comment of a referee's report on a work of ours (which we quote only as an extreme, but undoubtedly clear, exemplification of a widely diffused position about JP qp-s):

> The theory proposed by JP starts from the idea of an experimental *question*; the claim was made by JP that we could show *a priori* that the structure of the set of experimental *questions* was that of an *orthocomplemented lattice*. [This assertion seems to us quite curious, especially in making reference to the term "*a priori*"; at any rate, there is no trace of the aforesaid claim in any of JP's papers.] (· · ·) This claim has been severely criticized (see, e.g., Hughes, PSA 1982, Vol. I); the grounds of this criticism are that, if we accept the *operational definitions* given of the *meet* of two questions ($\alpha \wedge \beta$) and the *negation* of a question (α^v), then we find that
>
> $$\alpha \vee \beta =_{\text{def}} (\alpha^v \wedge \beta^v)^v = \alpha \wedge \beta \tag{5.1}$$
>
> which is an undesirable result, showing as it does that the operational definitions of meet and complementation cannot, via de Morgan's laws, define a proper lattice-theoretic join operation as they should [1st Referee's Report].

Our comment on this position, based on the foregoing discussion on the JP approach is that, if α, β are questions from $\langle \mathcal{Q}; I, \pi, ^v \rangle$, then: ^{v} must coincide with the only unary operation defined on $\mathcal{Q}$, i.e., the *inverse*, while $\wedge$ must coincide with the only nonunary operator defined on $\mathcal{Q}$, i.e., the *product* π.

In this case, the referee's equation

$$(\alpha^{\vee} \wedge \beta^{\vee})^{\vee} = \alpha \wedge \beta$$

is correct and well known in the framework of the JP theory; indeed, it is just

$$(\alpha^{\vee} \pi \beta^{\vee})^{\vee} = \alpha \pi \beta$$

which has been stated by Piron in a more general form, i.e., for any family $\{\alpha_i\}$ of questions [see (2.1) and (H.1) for the Hilbert space model],

$$(\pi \alpha_i^{\vee})^{\vee} = \pi \alpha_i$$

Therefore, the new operator $\vee$ defined in $\mathscr{Q}$ by (5.1) coincides with $\wedge$, i.e., is the product, and this does not introduce any "undesirable result," since $\mathscr{Q}$ is *not* a lattice and $\wedge$ is *not* a meet.

Indeed, in the JP approach no partial order relation is introduced on the set of questions Q; moreover, (DF2) or (D2) define only a quasi-order (a reflexive and transitive, but in general not an antisymmetric, relation) on Q and in JP's papers one cannot find any mention of some alleged partial ordering (and so, *a fortiori*, of a meet or an orthocomplementation) on the set of questions. Hence, only a superficial reading of JP's papers allows one to confuse the product with the meet and the inverse with the orthocomplementation with respect to some (not introduced) partial ordering on Q; consequently, the above $\vee$ operation cannot be interpreted as a lattice join and any reference to wrong behaviors with respect to some de Morgan law lacks significance.

From a different point of view, an empirical critique characterizes a report of a second referee, which is very interesting and is linked, in our opinion, always to the identifications between Piron questions and Mackey questions:

> Measurement of spin projections does not give yes–no information, when using a Stern–Gerlach device for example. Instead, if the particle is deflected up, there is a high probability that the spin was up, but not a probability = 1. In measuring screen observables, again one obtains only probabilistic results. In signal processing, in particles localization, and so on, *it is well established that the observables are based on effects which are operators with spectrum in* [0, 1], [i.e., operators such that $\mathbb{O} \leq F \leq \mathbb{1}$, our note], and which for most cases do not include 1 as an eigenvalue. Thus, *empirically speaking, one probably has few or no "questions" at all* [2nd Referee's Report].

With respect to this referee's comments, we do agree that "it is now well established that observables are based on *effects*," i.e., Hilbertian operators from $\mathscr{Q}(\mathscr{H})$, but in our Hilbert space model all effects are representatives of JP questions. Hence, the subsequent phrase "thus, empirically speaking, one probably has few or no questions at all" can have two possible interpretations.

(i) Either one interprets "questions" as JP questions, and in this case, empirically speaking, one probably has few or no "effects" at all, too.

(ii) Or, on the other hand, one interprets "questions" as M questions, and in this case it is correct that, empirically, there are probably few or no questions at all.

On the contrary, if according to our Hilbert space model, "effects" are representative of JP "questions" and if "effects" (at least a certain class of them) are "empirically definable," then the corresponding JP "questions" are "empirically definable," too, and at least there are as many questions as effects.

6. HTM CRITIQUE OF PIRON QP-S

A more subtle critique of the JP approach has been made by HTM based on the following "negative" results.

Theorem 6.1. In any JP qp-s the following statements hold.

$\mathscr{T}_2$. For every $a \in \mathscr{L} \backslash \{\mathbf{1}\}$, some $\alpha \in a$ exists such that $\alpha^{\vee} \notin a'$.

$\mathscr{T}_3$. For every $b, c \in \mathscr{L}$, $\beta \in b$, $\gamma \in c$ we have

$$\pi\{\beta, \gamma\} \in b \cap c \qquad \text{while} \qquad \pi\{\beta^{\vee}, \gamma^{\vee}\} \notin (b \cap c)'$$

Precisely, HTM say that the attempt by JP to produce a formal system of *propositions* in their early work (Piron, 1964; Jauch, 1968)

> has encountered a serious criticism: one of the axioms of the system, asserting the existence of a 'product proposition' $a \cap b$ for any pair (a, b) of propositions from the system, is devoid of semantic definability.
>
> Recently Piron has proposed a new formal system of "*questions*" and "*propositions*" claimed to eliminate this deficiency. Moreover, this system is claimed to be able to yield by interpretation quantum mechanics as well as any other known physical theory, thus offering a general syntactic scheme for physical theories of any kind.
>
> (· · ·) However, [the relevance of Piron's formal system] as a syntactic scheme for physical theories cannot be accepted. Indeed, by a succession of theorems it will be brought into evidence that one of the axioms of the system asserts the 'existence' of a class of propositions for which neither a syntactic method of construction is explicitly available inside the system, nor can a semantical definition be found in consistency with the semantic content assigned to the corresponding descriptive elements from the quantum mechanical formalism. Under such conditions it will be concluded that, as a syntactic scheme for the generation of quantum mechanics by interpretation, the formal system proposed by Piron so far has not attained its aim (Hadjisavvas *et al.*, 1980).

6.1. Concerning HTM Theorem $\mathscr{T}_2$

In particular, quoting HTM:

As far as we know, Axiom C has not yet been seriously criticized. We believe that this acceptance of Axiom C stems from a false assumption. Namely several authors (for example, Greechie and Gudder

Note. "Every proposition has at least one compatible complement. This can be seen as follows. If $\alpha \in a$ and let b be the equivalence class containing $\alpha^{\vee}$. Then b is *the* compatible complement of a" [Greechie and Gudder (1974)]

and, in an early version of qp-s, Jauch and Piron

"If $\alpha \in a$ is a yes–no experiment [i.e., a question] in the class a, then a' contains the experiment [i.e., the question] $\alpha^{\vee}$, which is the same experiment as α but with its alternative interchanged." [Jauch and Piron (1970)])

consider that the compatible complement of a' of a proposition of a qp-s simply consists of the class $\{\alpha^{\vee}\}$ of all the negations $\alpha^{\vee}$ of the questions $\alpha \in a$. [Of which Thieffine (1983) gives the following formalization:

Assumption U. The compatible complement a' of a proposition a of the formal system of questions and propositions consists of the class $\{\alpha^{\vee}: \alpha \in a\}$ of all inverses $\alpha^{\vee}$ of a question α, $\alpha \in a$.]

If this were true, Axiom C would be so trivially satisfied that the necessity of its statement as an axiom would be questionable. But in fact the mentioned assumption [i.e., U] is not true $(\cdots)$. This will be shown by $(\cdots)$

Theorem $\mathscr{T}_2$. For any $a \in \mathscr{L}$ distinct from the trivial proposition, the compatible complement a' is different from the class of the negations of all the questions of which a is the equivalence class.

$(\cdots)$ Thus—syntactically—we are already in the presence of the surprising fact that the negation of a given question lies somewhere outside the complement of the proposition to which that question belongs. $(\cdots)$ Thus contrary to a widely held opinion, the compatible complement a'—quite generally—cannot be formed as the class of all negations of the questions from a qp-s. *This, however, does not yet lead to doubt about the existence of* a'. Indeed so far there is still the possibility that for each given proposition a some method for constructing a nonvoid a' is specifiable, even if a' does not contain *all* the negations $\alpha^{\vee}$ of the $\alpha \in a$ (Hadjisavvas *et al.*, 1980).

As a consequence of the above discussion about the content of Theorem $\mathscr{T}_2$, HTM conclude that "what is questionable is the very "existence" of the proposition a', the compatible complement of any proposition a" (Hadjisavvas *et al.*, 1980).

In our opinion, the unique possible *negative content* of $\mathscr{T}_2$ is that it contradicts Assumption U in the presence of Axiom C, and nothing else. To explain this point better, and referring to our Hilbert space model, we, step by step in square brackets, comment upon the following discussion in Hadjisavvas *et al.* (1980).

Interpretative Illustration. Consider a particle in a one-dimensional space, say represented by the real line. Let α, β be two questions defined as follows: α is defined by an apparatus capable of verifying whether or not the position of the particle belongs to the set (0, 1), the answer 'yes' corresponding to the case that the particle is found in (0, 1); analogously, β is an apparatus capable of verifying if the position of the particle belongs to the set (2, 3)."

[In our $L_2(\mathbb{R})$ Hilbertian model, α is represented by any Hilbertian question $F(v_1)$, with $\Delta_1(v_1)=(0, 1)$, which is "true" in any preparation ψ such that $\mathrm{supp}(\psi)\subseteq(0, 1)$, and β by any Hilbertian question $F(v_2)$, with $\Delta_1(v_2)=(2, 3)$, which is "true" in any preparation φ such that $\mathrm{supp}(\varphi)\subseteq(2, 3)$.]

"Note that, by definition (PD2), α^v and β^v correspond to the same apparatus, but for these questions the answers 'yes' and 'no' have been inverted. Let a, b the propositions to which α and β belong."

[In the Hilbertian realization, $a=[E(0, 1)]_\sim$ and $b=[E(2, 3)]_\sim$ are the required propositions.]

"We note that in this illustration, apart from their syntactic definitions, the propositions a, b are furthermore endowed with a semantic content, for instance, here the proposition 'the position of the particle belongs to the space interval represented by (0, 1),' which can be verified or falsified by use of the apparatus α."

[Hilbertian proposition $a=[E(0, 1)]_\sim$ is associated to the Hilbertian localization property $l_q(0, 1)$, see (4.6), which gives the semantic content to a; property $l_q(0, 1)$ is actual in every preparation ψ such that $\mathrm{supp}(\psi)\subseteq(0, 1)$, and is potential in all other Hilbertian preparations.]

"By construction we have $\alpha\in a$, $\alpha^v\in a'$, $\beta\in b$, $\beta^v\in b'$. Furthermore, $b\subseteq a'$. Thus, b and a are orthogonal."

[This HTM assertion must be correctly interpreted. Precisely, if $\alpha\in a$, then one can construct $a^*(\alpha):=[\alpha^v]_\sim$, which depends on the question $\alpha\in a$ and is such that $\alpha^v\in a^*(\alpha)$; of course, a and $a^*(\alpha)$ satisfy condition (C2) of compatibility but, in general, condition (C1) does not hold for this pair of propositions and so $a^*(\alpha)\neq a'$, from which we get $\alpha^v\notin a'$. At any rate, Axiom C assumes that at least one $\alpha_1\in a$ exists such that $a'=a^*(\alpha_1)$ is the compatible complement of a; owing to Axiom P, this compatible complement is unique (incidentally, the HTM proof of $\mathscr{T}_2$ refers just to this α_1). In our Hilbertian model, the unique compatible complement of $a=[E(0, 1)]_\sim$ exists and is $a'=[E((0, 1)^c)]_\sim$. For any $\alpha=F(v_1)\in[E(0, 1)]_\sim$, different from $E(0, 1)$, we have $F(v_1)'\in[E(\Delta_0(v_1))]_\sim$; hence, $a^*(\alpha)=[E(\Delta_0(v_1))]_\sim\neq a'$ and $F(v_1)'\notin[E((0, 1)^c)]_\sim=a'$. An analogous result holds for $b=[E(2,3)]_\sim$, $b'=[E((2,3)^c)]_\sim$, and $F(\nu_2)\in[E(2,3)]_\sim$. Note that in general we cannot affirm that $b\subseteq a^*(\alpha)$; on the contrary, $b\subseteq a'$ and $\forall\alpha\in a$, $a^*(\alpha)\subseteq a'$.]

"If we now define the question $\gamma=\alpha\pi\beta^\nu$, then, as shown in the proof of Theorem $\mathscr{T}_2$, one has $\gamma\in a$, $\gamma^\nu\notin a'$."

[Bearing in mind (4.5), for the Hilbertian localization questions discussed above we get

$$\gamma=F(\nu_1)\Pi F(\nu_2)'=\tfrac{1}{2}(\chi_{((0,1)\cap\Delta_0(\nu_2))}+\chi_{((2,3)\cap\Delta_0(\nu_1))^c})(\cdot)$$

but if condition $(0,1)=\Delta_0(\nu_2)$ does not hold, then we cannot state that $\gamma\in a$. On the other hand, the Hilbertian localization question

$$\gamma_1=E(0,1)\Pi E(2,3)'=(\chi_{(0,1)}+\tfrac{1}{2}(\chi_{((-\infty,0]\cup[1,2]\cup[3,+\infty))}))(\cdot)$$

is such that $\gamma_1\in a$, $\gamma_1^\nu\notin a'$.]

In conclusion, Axiom C ensures that for any proposition a a compatible complement a' exists; in particular, as a consequence of C_2, a question $\alpha_1\in a$ exists such that $\alpha_1^\nu\in a'$. But, in general, for any other $\alpha\in a$, $\alpha\neq\alpha_1$, $\alpha^\nu\in a^*(\alpha)\neq a'$. This result can be summarized by the following general statement (Cattaneo *et al.*, 1988)

$$\alpha\sim\alpha_1 \quad \text{does not imply} \quad \alpha^\nu\sim\beta^\nu \tag{6.1}$$

that is,

$$[\alpha]_\sim=[\alpha_1]_\sim \qquad \text{while, in general} \qquad [\alpha^\nu]_\sim\neq[\alpha_1^\nu]_\sim \tag{6.2}$$

[In the $L_2(\mathbb{R})$ localization example, $F(\nu_1)\sim F(\nu_2)$ iff $\Delta_1(\nu_1)=\Delta_1(\nu_2)$, while $F(\nu_1)'\sim F(\nu_2)'$ iff $\Delta_0(\nu_1)=\Delta_0(\nu_2)$; the latter conditions does not hold in general.]

Remark 1. In the light of our Hilbertian model, we find interesting the following answer of Foulis and Randall (1984) to HTM's critique: "In a generalized qp-s it is quite possible for a proposition to have more than one compatible complement, in conformity with the words of Greechie and Gudder. Greechie and Gudder do not say that the compatible complement of a' consists of the class $\{\alpha^\nu\colon \alpha\in a\}$ nor even that the latter set is an equivalence class at all. What they assert is that, for each $\alpha\in a$, the equivalence class $b=\{\alpha^\nu\}$ is a compatible complement of a—and this is a quite different matter."

As we have already noticed, if $F(\nu)$ is a Hilbertian localization question with associated proposition $[F(\nu)]_\sim=[E(\Delta_1(\nu))]_\sim$, then

$$[F(\nu)']_\sim=[E(\Delta_0(\nu))]_\sim$$

in general does not satisfy condition (C1) and so the FR assertion is not true. At any rate, observing that Greechie and Gudder do not assert that "b is a compatible complement of a," rather that "b is *the* compatible complement of a," for the sake of completeness, we also quote the reply of Hadjisavvas and Thieffine (1984): "Immediately after the cited note, Greechie and Gudder state Axiom P, which is valid for Piron's qp-s and which *implies uniqueness of the compatible complement.* It follows from this uniqueness and from their note that the compatible complement of a proposition a contains all a^{v} for $\alpha \in a$."

6.2. Concerning HTM Theorem $\mathscr{T}_3$

We now quote another critique by HTM of Axiom C, based on their Theorem $\mathscr{T}_3$.

Let us consider any pair of distinct propositions $a \in \mathscr{L}$, $b \in \mathscr{L}$, $a \neq b$, $a = \{\alpha_i\}$, $b = \{\beta_j\}$.

Let us consider all the product questions $\alpha_i \pi \beta_j$, $\alpha_i \in a$, $\beta_j \in b$.

As a consequence of Theorem $\mathscr{T}_3$, "now we assert that the compatible complement $(a \cap b)'$ of $a \cap b$ contains *none* of the negations $(\alpha_i \pi \beta_j)^{v}$ of the question $(\alpha_i \pi \beta_j)$ by help of which—exclusively—the proposition $a \cap b$ can be defined. $(\cdots)$

The content of Theorem $\mathscr{T}_3$ can be expressed graphically as follows:

$$\begin{array}{cc} (a, b) \in \mathscr{L} \times \mathscr{L}, a \neq b & (\alpha\pi\beta) \in a \cap b \\ \updownarrow & \updownarrow \\ \alpha \in a, \beta \in b & (\alpha\pi\beta)^{v} \notin (a \cap b)' \text{"} \end{array}$$

(Hadjisavvas *et al.*, 1980).

As to this critique, we do agree with HTM when they assert that "inside the qp-s, each question α defines (as its equivalence class) a certain proposition a" (Hadjisavvas *et al.*, 1980), but it must be stressed that it does not—*exclusively*—define this proposition a; rather, in a certain very specific situation, a question may be privileged to determine a peculiar proposition [for instance, as a consequence of (L1) Theorem 4.1, question $\alpha\pi\beta$, $\alpha \in a$, $\beta \in b$, can be chosen to determine $a \cap b$], but, in accordance with Foulis and Randall (1984), "the equivalence class $(a \cap b)$ is determined by any question γ in $(a \cap b)$, and γ needs not have the form $\alpha\pi\beta$."

As a consequence of Theorem $\mathscr{T}_3$, HTM set the following:

Illustrative Challenge:

—Let then $a \in \mathscr{L}$ and $b \in \mathscr{L}$ be, respectively, two propositions defined by two chosen questions α and β.

—Each pair of propositions from $\mathscr{L}$ defines a corresponding product proposition belonging to $\mathscr{L}$

[(i) form $\alpha\pi\beta$ using (a3), Section 3, and the usual rules of formation of wffs] so in particular one can write $a \wedge b \in \mathscr{L}$

[(ii) form $[\alpha\pi\beta]_{\sim}$ using (D4), which, by (D5), belongs to $\mathscr{L}$; owing to (L1), Theorem 3.1, we have that $[\alpha\pi\beta]_{\sim} = a \wedge b$].

—Finally, Axiom C asserts the existence inside $\mathscr{L}$ of the compatible complement p' of any proposition $p \in \mathscr{L}$, hence, also in particular, according to Axiom C, $(a \cap b)' \in \mathscr{L}$ does exist.

[(iii) By (C1) Axiom C a compatible complement $(a \cap b)'$ for $a \cap b$ exists in $\mathscr{L}$; moreover, by (C2) Axiom C, at least a question $\gamma \in Q$ exists such that $\gamma \in (a \cap b)$ and $\gamma^{v} \in (a \cap b)'$.]

—But according to Theorem $\mathscr{T}_3$ none of the negations $(\alpha_i \pi \beta_j)^{v}$ of the questions $\alpha_i \pi \beta_j$, $\alpha_i \in a$, $\beta_j \in b$ does belong to $(a \cap b)'$.

Our challenge then is the following one:

1. Try to specify syntactically inside the qp-s at least one question belonging to $(a \cap b)'$ $(\cdot\cdot\cdot)$.
2. Specify the apparatus corresponding to this question (Hadjisavvas *et al.*, 1980).

The answer to point 1 of the challenge is given by (i)–(iii) in the above square brackets, which are obtained by a pure syntactic construction inside JP qp-s formalized theory. Since the (C2) of Axiom C postulates the existence of at least one question γ such that $\gamma \in (a \cap b)$ and $\gamma^{v} \in (a \cap b)'$, which is supposed, according to (BC1) of Section 2, to correspond to a well-defined "measuring apparatus," point 2 of this challenge is devoid of sense. The very content of (C2) Axiom C consists just in "postulating the existence of the measurement γ (and of the corresponding apparatus)" (Foulis and Randall, 1984) and axioms are introduced in a formal system just to postulate something.

Remark 2. An exemplification of the above "illustrative challenge" can be given using Hilbertian localization questions of Section 4.2.

Let $\alpha = F(v_1) \in [E(0,1)]_{\sim} = a$ and $\beta = F(v_2) \in [E(2,3)]_{\sim} = b$ be the two Hilbertian localization propositions $l_q(0,1)$ = "the particle is (quantistically) localized in (0, 1)" and $l_q(2,3)$ = "the particle is (quantistically) localized in (2, 3)," unsharply tested by the Hilbertian localization questions $F(v_1)$, $\Delta_1(v_1) = (0,1)$, and $F(v_2)$, $\Delta_1(v_2) = (2,3)$, respectively.

The product Hilbertian localization question, according to (4, 5), is

$$F(v_1)\Pi F(v_2) = \tfrac{1}{2}(\chi_{(\Delta_0(v_1)^c \cup \Delta_0(v_2)^c)})(\cdot)$$

[where $\Delta_0(v_1)^c \cup \Delta_0(v_2)^c \subseteq (-\infty, 0] \cup [1,2] \cup [3, +\infty)$, whose corresponding Hilbertian localization proposition is $a \cap b = [F(v_1)\Pi F(v_2)]_{\sim} = \mathbf{O}$, i.e., the absurd proposition.

The compatible complement of $a \cap b = \mathbf{O}$ exists and is $(a \cap b)' = a' \cup b' = \mathbf{1}$; indeed, (C1) $\mathbf{O} \cap \mathbf{1} = \mathbf{O}$ and $\mathbf{O} \cup \mathbf{1} = \mathbf{1}$; (C2) $E(\varnothing) = \mathbb{O} \in \mathbf{O}$ and $E(\varnothing)' = E(\varnothing^c) = \mathbb{1} \in \mathbf{1}$. [where $\gamma = E(\varnothing)$].

But

$$(F(v_1)\Pi F(v_2))' = (1 - \tfrac{1}{2}\chi_{((\Delta_0(v_1)^c \cup \Delta_0(v_2)^c)})(\cdot)$$

and none of these Hilbertian localization questions belongs to **1**, in agreement with Theorem $\mathscr{T}_3$.

In conclusion, from Theorem $\mathscr{T}_3$ one does not show that Piron's Axiom C is devoid of any interpretation; this theorem shows only the impossibility of constructing proposition $(a \cap b)'$ starting from negations $(\alpha\pi\beta)^v$ of any $(\alpha\pi\beta) \in (a \cap b)$, and nothing else.

7. THE PHYSICAL CONTENT OF AXIOMS C AND A_1

In Section 6.2 we saw that, contrary to HTM's opinion, syntactically a method of construction of the "product proposition" according to (i)–(iii) (square brackets of the illustrative challenge) is available inside JP qp-s.

On the other hand, people working in QP might have sufficient physical motivations to reject the physical content, i.e., the very existence, of question γ in (C2), for instance, because it does not "correspond to a fundamental physical fact" (Hadjisavvas and Thieffine, 1984). In the present section we will try to give a content to Axiom C, from a physical point of view, in such a way that a "semantical definition" of product proposition, e.g., the one illustrated in Remark 2 for Hilbertian localization questions, is given in terms of unsharp tests of propositions; this semantical definition is "in consistency with the semantic content [i.e., unsharp measurement of propositions] assigned to the corresponding descriptive elements [i.e., Hilbertian fuzzy questions in $Q(\mathscr{H})/(\mathscr{E}(\mathscr{H})$] from the quantum mechanical formalism [i.e., the standard orthodox Hilbert space quantum mechanics in $\mathscr{H}$, as outlined in Section 4]" (Hadjisavvas *et al.*, 1980).

Thus we now introduce the following axiom, which will turn out to be stronger than Piron's Axiom C.

Axiom CC. $\forall \alpha \in Q$, $\exists \Box\alpha \in Q$ such that the following statements hold:

(dq-1) $\alpha \sim \Box\alpha$
(dq-2) $\beta \sim \Box\alpha$ implies $F(x, \beta) \Rightarrow F(x, \Box\alpha)$
(dq-3) $\Box\alpha \sim \Box\beta$ implies $\Box\alpha = \Box\beta$
(dq-4) $\gamma \sim (\Box\alpha)^v$ implies $F(x, \gamma) \Rightarrow F(x, (\Box\alpha)^v)$

Questions $\Box\alpha$, for α running in Q, are said to be *ideal questions* and their collection is denoted by

$$\mathscr{E} := \{\Box\alpha : \alpha \in Q\}$$

Remark 1. For any $\alpha \in Q$, the element $\Box\alpha \in Q$ assured by Axiom CC is unique; indeed, if $(\Box\alpha)'$ and $(\Box\alpha)''$ are two such elements, then from (dq-1) it follows that $(\Box\alpha)' \sim (\Box\alpha)''$ and from this and (dq-3) we get that $(\Box\alpha)' = (\Box\alpha)''$. Note that the following are immediate consequences of Axiom CC:

$$(\forall \alpha \in Q), ([\alpha]_\sim = [\Box\alpha]_\sim) \tag{7.1}$$

$$(\forall \alpha \in Q), (\Box\Box\alpha = \Box\alpha) \tag{7.2}$$

$$(\forall e \in \mathscr{E}), (\Box e = e) \tag{7.3}$$

In the case of a JP qp-s with Axiom CC, the syntactic scheme of the qp-s can be endowed with another 1-argument functor $\Box: Q \mapsto Q$, "*necessity*," and the formal structure of qp-s can be summarized in the following way:

$$\mathbf{L}_{\mathrm{JP}}^{(\mathrm{CC})} \equiv \langle S, Q, \mathscr{P}(Q); I, {}^\vee, \pi, \Box; T \rangle$$

Of course,

$$\alpha \sim \beta \qquad \text{implies} \qquad \Box\alpha = \Box\beta$$

which allows a one-to-one correspondence between propositions and ideal questions, pictured in the following graph:

$$\begin{array}{c} Q \ni \alpha \\ \downarrow^{\Box} \\ [\Box\alpha]_\sim \in \mathscr{L} \;\equiv\; \mathscr{E} \ni \Box\alpha \end{array} \tag{7.4}$$

For any question $\alpha \in Q$, let $a = [\alpha]_\sim \in \mathscr{L}$ be the proposition generated by α; then, (1) owing to (dq-1), the ideal question $\Box\alpha$ belongs to a, (2) it minimalizes, owing to (dq-2), the randomness of the certainly-false domains of all other questions from a, and, (3) owing to (dq-3) it is the *unique* question from a with these properties.

Thus, for these reasons, $\Box\alpha$ is said to be the *exact* representative of the proposition a, while all other questions from a are called *fuzzy* representatives of the same proposition a. In conclusion, $\Box\alpha$ is the (unique) ideal question sharply *testing* proposition a (and all other questions from a test the latter in an unsharp way).

The role of condition (dq-4) is linked to the following results.

Theorem 7.1. The mapping

$$\mathscr{L} \mapsto \mathscr{L}, \quad a = [\alpha]_\sim \to a' := [(\Box\alpha)^\vee]_\sim$$

is such that the structure

$$\langle \mathscr{L}, \mathbf{0}, \leq, ' \rangle$$

is an orthocomplemented complete lattice, which satisifes the JP condition C. Indeed,we have:

(C1) $a \wedge a' = \mathbf{O}$ and $a \vee a' = \mathbf{1}$.
(C2) $\Box \alpha \in a$ and $(\Box \alpha)^{\vee} \in a'$.

In a certain sense, Axiom CC gives the positive content of Axiom C, consisting in describing a peculiar physical entity in which any proposition, and so the corresponding property, can be tested by exactly one *exact* (or *sharp*) measurement, besides a suitable collection of *fuzzy* (or *unsharp*) questions which test with noise and imprecision the same property.

Referring to the scheme of Section 5, and taking into account (7.4), we have the following complete diagram which summarizes the most important behaviors of JP qp-s with Axiom CC:

$$\begin{array}{ccc} Q & \xrightarrow{\mathscr{H}\text{-real}} & Q(\mathscr{H}) \\ \Box\downarrow & & \downarrow\Box \\ \mathscr{L} \equiv \mathscr{E} & \xrightarrow{\mathscr{H}\text{-real}} & \mathscr{E}(\mathscr{H}) \equiv \mathscr{M}(\mathscr{H}) \equiv \mathscr{L}(\mathscr{H}) \end{array}$$

As one can see from the above diagram, the Hilbert space model of qp-s we outlined in Section 4 satisfies Axiom CC since, owing to (4.1), for any Hilbertian proposition $[F]_{\sim}$, characterized by the common closed subspaces $M_1(F)$ of preparations (nonzero vectors) in which the proposition is true, the orthogonal projection $E_{M_1(F)}$ exists and satisfies conditions (dq-1)–(dq-4).

In this way, orthogonal projections from $\mathscr{E}(\mathscr{H})$ are the ideal or exact questions of our Hilbert space model of qp-s; they sharply test the Hilbertian propositions, whereas all other Hilbertian questions from $Q(\mathscr{H})/\mathscr{E}(\mathscr{H})$ are representatives of fuzzy or unsharp measurements of the involved propositions.

Remark 2. As an application, let us consider the $L_2(\mathbb{R})$ Hilbertian localization questions. For any Borel subset Δ of $\mathbb{R}$ we have the Hilbertian localization proposition $a(\Delta) = [E(\Delta)]_{\sim}$ with corresponding quantum localization property $l_q(\Delta) =$ "the particle is localized in Δ" [recall that, owing to (7.3), $\Box E(\Delta) = E(\Delta)$]. The question $E(\Delta)$ sharply tests the proposition $a(\Delta)$; it is the unique Hilbertian sharp localization question which tests this proposition; all the Hilbertian localization questions $F(\nu)$, with $\Delta_1(\nu) = \Delta$, test in an unsharp way the same proposition. We can introduce the necessity mapping, whose restriction to the Hilbertian localization questions acts in the following way:

$$F(\nu) \mapsto \Box F(\nu) = E(\Delta_1(\nu))$$

As to the orthocomplementation, we have that if

$$a(\Delta) = [E(\Delta)]_\sim = [F(\nu)]_\sim, \qquad \text{with} \qquad \Delta_1(\nu) = \Delta$$

Then

$$a(\Delta)' = [E(\Delta^c)]_\sim = [F(\nu^*)]_\sim, \qquad \text{with} \quad \Delta_1(\nu^*) = \Delta^c$$

Of course,

$$F(\nu) \in a(\Delta) \text{ and } F(\nu)' \in a(\Delta)' \qquad \text{iff} \qquad \Delta_1(\nu) = \Delta \text{ and } \Delta_0(\nu) = \Delta^c$$

and this happens, iff $F(\nu) = E(\Delta)$. This result agrees with the aforegoing discussion about Axiom C and Assumption U: $a(\Delta)$ and $a(\Delta)'$ are compatible complements tested by the sharp localization questions $E(\Delta)$ and $E(\Delta)' = E(\Delta^c)$, respectively, but for any $F(\nu) \in a(\Delta)$, with $\nu \neq \chi_\Delta$, we have that $F(\nu)' \notin a(\Delta)'$.

7.1. Pure State Property and Axiom A_1

We now discuss the role of Axiom A_1 with respect to the pure state property.

Definition 7.1. A qp-s is said to have the pure states property (briefly, to be a PS qp-s) iff the following axiom holds.

Axiom PS1. For every $x \in S$, some $x_p \in S_p$ exists such that $\sigma(x) \subseteq \sigma(x_p)$.
Axiom PS2. For every $x \in S$,

$$\sigma(x) = \bigcap_{\sigma(x) \subseteq \sigma(x_p)} \sigma(x_p)$$

Let us comment briefly on Axiom PS1 from a physical point of view. As seen in (D20), any pure preparation x_p is a preparation such that $\sigma(x_p)$ is maximal in Σ, ordered by set inclusion, so that $\sigma(x_p)$ embodies a "maximum amount of information" about the actual properties of the physical system prepared in x_p. Axiom PS1 states that for any preparation x the set $\sigma(x)$ is contained in at least one (in general not unique) of these maximal subsets of propositions (or actual properties).

Thus, whenever x_p is a pure preparation, the available amount of information coincides with a maximum of information which can be attributed to any individual sample of the physical system prepared in x_p. A position of this kind is found in Jauch and Piron (1969): "A state of a system is the set σ of all true propositions of the system [i.e., (D16)] $(\cdots)$. We may think of the state as containing the maximal amount of information that is possible concerning an individual system [i.e., the state is a pure state; see (D18)]. Thus we shall *postulate* that two states σ_1 and σ_2 cannot be subsets of one another."

Remark 3. As a consequence of the latter postulate, in the JP approach all states are pure and thus it is correct to assert that: "We shall in fact *assume* that *every* individual system, be it an isolated system or a member of a statistical ensemble, is in a definite state as defined above" (Jauch and Piron, 1969).

In our modified approach, according to Definition 7.1, we consider also the possibility of nonpure states.

If preparation x is not pure, the available amount of information is only a part of the maximal amount of information embodied in every maximal subset $\sigma(x_p) \in \Sigma_p$ which contains $\sigma(x)$; but it must be stressed that the knowledge of $\sigma(x)$ alone does not allow any privileged choice among these maximal subsets. We observe that one can claim (or assume) that also in this case to every individual sample of the physical system prepared according to x one of $\sigma(x_p)$, i.e., a pure state, containing $\sigma(x)$ can be attributed in any case; yet, this assumption sounds rather metaphysical.

Remark 4. Note that if a pure state is attributed to any individual sample of a physical system prepared according to x, in the case in which x is not pure, this state may change from one sample to another and must not be confused with the information which is actually available about the sample [which is the same for every sample and is summarized in $\sigma(x)$].

To be precise, if x is any nonpure preparation whose corresponding state is $\sigma(x)$, the latter being collection of all the actual properties of any single sample prepared according to x (whether these properties have been measured), a certain number of pure states

$$\Sigma_p(x) := \{\sigma(x_p) : x_p \in \Sigma_p,\ \sigma(x) \subseteq \sigma(x_p)\}$$

is associated to x.

If $\{i_j : j \in J\}$ is a set of individual samples prepared according to x, then to any of such sample i_j a pure state $\sigma^{(i_j)}(x_p) \in \Sigma_p(x)$ can be attributed. But from the fact that the set $\Sigma_p(x)$ of all pure states associated to x is not a singleton, this state may change with the choice of sample i_j, in agreement with the following interpretation of the JP assertion: "it is important to distinguish the [pure] state [$\sigma^{(i_j)}(x_p) \in \Sigma_p(x)$] of a [sample i_j of the] system [prepared in x] from the amount of information [$\sigma(x)$] available about the [same sample i_j of the] system" (Jauch and Piron, 1969).

Let us consider now the more relevant consequences of Axiom PS, whose proofs are in Cattaneo *et al.* (1989).

Proposition 7.1. In any PS pq-s, Axiom A_1 follows from PS1. Moreover, from PS2 we get the set of all pure states is order determining the set of propositions, i.e., recalling (D26),

$$a \subseteq b \quad \text{iff} \quad \forall u_p \in \Sigma_p, \mathscr{A}(u_p, a) \quad \text{implies} \quad \mathscr{A}(u_p, b)$$

8. PRE-HILBERT MODEL OF WEAK QP-S

We conclude by presenting a pre-Hilbert model of a question–proposition system, in which basic specific Axiom 3 is restricted to any finite number of questions only; moreover, neither Axiom CC nor thus Axiom C of the JP approach holds. This pre-Hilbert JP-like mathematical structure is a model of a qp-s describing a physical entity in which some of the JP strong axioms can be weakened if some physical motivations lead one to reject them.

Let $\mathscr{S}$ be a complex, in general separable, pre-Hilbert space. Define $S(\mathscr{S}) := \mathscr{S}/\{\underline{0}\}$, the set of all preparations, and $Q(\mathscr{S}) := \{F: \mathscr{S} \mapsto \mathscr{S} \mid \text{linear}, \mathbb{O} \leq F \leq \mathbb{1}\}$, the set of all questions of a pre-Hilbert realization of a qp-s, where the certain question is the identity $\mathbb{1}$; the inverse question of F is $F' = \mathbb{1} - F$; the product of F_1 and F_2, denoted by $F_1 \Pi F_2$, is $\frac{1}{2}(F_1 + F_2)$.

Analogously to the Hilbert space case, for any $F \in Q(\mathscr{S})$ we introduce the subspaces $M_1(F) := \text{Ker}(\mathbb{1} - F)$ and $M_0(F) = \text{Ker}(F)$; the certainly-true domain and the certainly-false domain of F are defined as $S_1(F) := M_1(F)/\{\underline{0}\}$ and $S_0(F) := M_0(F)/\{\underline{0}\}$, respectively. We define the binary relations $T_{\mathscr{S}}(\psi, F) \Leftrightarrow \psi \in M_1(F)$ and $F_{\mathscr{S}}(\psi, F) \Leftrightarrow \psi \in M_0(F)$, interpreted as "question F is true (resp., false) in preparation ψ."

It is straightforward to verify that we have constructed a pre-Hilbert realization of a question–proposition formal language with Axiom 3 restricted to pairs of questions only. In particular, $F < G$ iff $M_1(F) \subseteq M_1(G)$ and so $F \sim G$ iff $M_1(F) = M_1(G)$.

The set of all propositions is identifiable with the set of all exact or $\perp$-closed subspaces of $\mathscr{S}$ (Cattaneo and Marino, 1986; Dvurecenskij, 1988), $\mathscr{M}(\mathscr{S}) = \{M \subseteq \mathscr{S}: M = M^{\perp\perp}\}$, which is an orthocomplemented complete lattice, which is not orthomodular (otherwise $\mathscr{S}$ would be a Hilbert space). To any proposition $[F]_{\sim} = \{F: F \sim G\}$ we can associate the unique certainly-true domain $S_1(F)$, the satisfaction domain of the proposition.

The structure $\langle \mathscr{S}, \mathscr{F}(\mathscr{S}); \mathbb{1}, ', \Pi; t \rangle$ does not satisfy Axiom CC (unless $\mathscr{S}$ is a Hilbert space) since a linear pre-Hilbertian question $F \in Q(\mathscr{S})$ admits the ideal question $\square(F)$ iff $\square(F)$ is an orthogonal projection, and this occurs iff $M_1(F)$ is a splitting subspace, i.e., iff $M_1(F) \oplus M_1(F)^{\perp} = \mathscr{S}$ (every splitting subspace is an exact subspace, too; the converse characterizing Hilbert spaces).

Therefore, in the pre-Hilbert space case we have some propositions which are of the *first kind*, precisely those propositions $[F]_{\sim}$ for which $M_1(F)$ is a splitting subspace and the corresponding *ideal question* is the orthogonal projection $E_{M_1(F)}$ and some propositions $[G]_{\sim}$ which are of the *second kind*, corresponding to a subspace $M_1(G)$ which is exact but not splitting.

In conclusion, the strict pre-Hilbert space example is a realization of a qp-s which does not satisfy Axiom CC; the set of all propositions is an orthocomplemented complete lattice which is not orthomodular. Only the first kind of propositions are measured by ideal questions, represented by orthogonal projections, and the set of all first-kind propositions is an orthomodular orthoposet. The Hilbert space case corresponds to a pre-Hilbert qp-s with Axiom CC; in this case the necessity modal-like operator associates to any question F the ideal question, i.e., the orthogonal projection $E_{M_1(F)}$, which exactly tests the property $[F]_{\sim}$.

Example 8.1. Let us consider the pre-Hilbert space $\mathscr{S}(\mathbb{R})$ of rapidly decreasing test functions (i.e., infinitely differentiable complex functions which, together with all their derivatives, vanish as $|x| \to \infty$ faster than the reciprocal of any polynomial). For any infinitely differentiable classical localization mapping $v: \mathbb{R} \mapsto [0, 1]$, the linear operator $F(v): \mathscr{S}(\mathbb{R}) \mapsto \mathscr{S}(\mathbb{R})$, $\psi \to v \cdot \psi$ is well defined and represents an $\mathscr{S}(\mathbb{R})$-localization question, which localizes particles in the Borel subset $\Delta_1(v)$. These pre-Hilbertian localization questions are of the second kind because the pre-Hilbert space $\mathscr{S}(\mathbb{R})$ is not invariant with respect to the linear operator

$$\psi \to \chi_{\Delta_1(v)} \cdot \psi$$

9. CONCLUDING REMARK

As a final consideration, we hope that those with sufficient metatheoretical or empirical reasons to reject some specific axioms of the theory can suggest how to modify the structure in some essential points (or to reject the whole theory). But, in any case, further discussions should make an appropriate use of the involved terms avoiding critiques based on misunderstandings.

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